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PETROV-GALERKIN METHOD FOR APPROXIMATION OF SOLUTIONS TO OPERATOR EQUATIONS IN POSITIVE AND NEGATIVE BANACH SPACES.

W.G. LITVINOV

Abstract. The Galerkin method, in particular, the Galerkin method with finite elements (called finite element method) are widely used in numerical solving of differential equations. The Galerkin method allows us to obtain approximations of weak solutions only. However, a rich variety of problems arise in applications, where approximations of smooth solutions and solutions in negative spaces have to be found. This paper is devoted to the employment of the Petrov-Galerkin method for solving such problems. General results on convergence of the Petrov-Galerkin approximations of solutions to operator equations are obtained. The problem on construction of the subspaces, which ensure the convergence of the approximations, is investigated. By way of example, we consider two–and–three-dimensional problems of the elasticity, a parabolic problem, and a nonlinear problem of the plasticity.

Key words.
Approximation, positive space, negative space, projection, convergence, kernel.

1. Introduction

Let U be a separable reflexive Banach space and U_k be a finite-dimensional subspace of U. Let A be a mapping from U to the dual space U^* of U.

Consider the problem: Find $u \in U$ such that

$$A(u) = 0. \tag{1.1}$$

A function $u_k \in U_k$ that satisfies the condition

$$(A(u_k), h) = 0, \quad h \in U_k, \tag{1.2}$$

is said to be the Galerkin approximation of a solution u to the equation (1.1).

Once $m(k)$ is the dimension of U_k, and $\varphi_1, \varphi_2, \dots, \varphi_{m(k)}$ is a basis of U_k, we have $u_k = \sum_{i=1}^{m(k)} a_i\varphi_i$, and (1.2) is represented in the form

$$\Big(A(\sum_{i=1}^{m(k)} a_i\varphi_i), \varphi_j\Big) = 0, \quad j = 1, 2, \dots, m(k). \tag{1.3}$$

The Galerkin method and the finite element approximation, that originates from the Galerkin method, are widely used for solving numerically and exploring the solvability of differential equations.

However, the condition A is a mapping from U to U^* is a strong restriction on A. The Galerkin method allows us to approximate weak solutions of elliptic equations. The Faedo-Galerkin method, that is a modification of the Galerkin method, which reduces the original problem to a system of ordinary differential equations, is used for approximation of weak solutions of nonstationary problems.

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On approximation of smooth solutions to differential equations and solutions of differential equations in negative spaces, we consider that A maps the space U to a space V other than U^*, see [10, 17]. The Galerkin method is unusable for such problems.

Problems on approximation of smooth solutions and solutions in negative norms appear, in particular, in the theory of optimal control for partial differential equations, see [12]. The investigation of the strength of a structure involves the computation of values of a function of the components of the stress tensor, see e.g. [21]. The values of this function should not exceed some magnitude at each point of the domain occupied with the structure. Since the stresses are defined by derivatives of the function of displacement, smooth solutions should only be used in such problems.

The optimization of the structure is associated with the searching of a solution to a conjugate problem in a negative space.

Where the operator A maps the space U into a space V, the Petrov-Galerkin method can be used advantageously. Let U_k and V_k^* be k-dimensional subspaces of U and V^*, V^* being the dual space of V.

A function $u_k \in U_k$ that satisfies the condition

$$(A(u_k), h) = 0, \quad h \in V_k^*, \tag{1.4}$$

is said to be the Petrov-Galerkin approximation of a solution u to the problem $A(u) = 0$.

In the special case that $V = U^*$, we can take $V_k^* = U_k$. Then the Petrov-Galerkin method is transformed into the Galerkin method.

Below in Section 2, we consider the case where the operator A is a linear continuous and one-to-one mapping of U onto V. A statement on the convergence of the Petrov-Galerkin approximations u_k in U to the solution u of the problem $Au = f$ is proved. In Section 3, we treat a problem on construction of the subspaces U_k and V_k^*, which ensure the convergence of the Petrov-Galerkin approximations in the case that, A is a one-to-one mapping of U onto V. The convergence of the Petrov-Galerkin approximations in the case, where the kernel and the defect of A are finite-dimensional subspaces of U and V, is investigated in Section 4. In Section 5, we consider two-and-three-dimensional problems of the elasticity theory and prove the existence of solutions to these problems in the space $W_p^2(\Omega)^n$, where $p \geq 2$, $n = 2$ or 3, and in $W_p^{2+\alpha}(\Omega)^n$, $\alpha \in (0, 1)$. The Petrov-Galerkin method for the problems of the elasticity theory is expounded in Section 6. In Section 7, the Petrov-Galerkin method is applied to the approximation of the solution of a parabolic problem.

In Section 8, we consider a class of nonlinear operator equations, whose operators are restrictions of operators which map Banach spaces to Banach spaces in a one-to-one manner. In particular, restrictions of strongly monotone operators are contained in this class.

Under some assumptions, we prove existence and uniqueness of smooth solutions to these operator equations, and the convergence of the Petrov-Galerkin approximations to the exact solutions. Here the approximations are computed by the modified Newton method, and in doing so, we change the spaces V_k^* in the Newtonian iterations.

By way of example, we consider in Section 9 approximation of the smooth solution to a nonlinear problem of the plasticity theory.

We mention that the Petrov-Galerkin method for linear elliptic operators in positive Hilbert spaces was considered in [3].

2. **Convergence of the Petrov-Galerkin approximations I.**

We use the following notations: We denote the space of linear continuous operators mapping a normed space X to a normed space Y by $\mathcal{L}(X,Y)$. If $A \in \mathcal{L}(X,Y)$, the adjoint of A operator is symbolized by A^*, and the inverse of A, if it exists, by A^{-1}. The dual of X space is denoted by X^*, and by (h,f) the duality between X and X^*, where $h \in X$ and $f \in X^*$. In particular, if $f \in L_2(\Omega)$ or $f \in L_2(\Omega)^n$, then (h,f) is the scalar product in $L_2(\Omega)$ or in $L_2(\Omega)^n$, respectively. If y is a point of X and G a set of X, we denote the distance between y and G by $\rho(y,G)$,

$$\rho(y,G) = \inf_{z \in G} ||y - z||_X.$$

$\mathbb{R}_+$ is the set of nonnegative numbers. The sign $\rightharpoonup$ denotes weak convergence in a Banach space. If M is a subspace of a normed space X, we denote by X/M the quotient space of X by M.

We assume that the following conditions are satisfied:

(C2.1): U and V are separable reflexive Banach spaces.

(C2.2): A is a linear continuous and one-to-one mapping of U onto V.

Let $\{U_k\}$, $\{V_k\}$, and $\{V_k^*\}$ be sequences of finite-dimensional subspaces of U, V, and V^* such that

$$\lim_{k\to\infty} \inf_{h \in U_k} ||v - h||_U = 0, \quad v \in U, \tag{2.1}$$

$$V_k = A(U_k), \quad V_k^* = (A(U_k))^*. \tag{2.2}$$

Let f be a given function of V. It follows from $(C2.2)$ that there exists a unique u satisfying

$$u \in U, \quad Au = f. \tag{2.3}$$

The Petrov-Galerkin approximations u_k of u are defined by

$$u_k \in U_k, \quad (Au_k, h) = (f, h), \quad h \in V_k^*. \tag{2.4}$$

Let $m(k)$ be the dimensions of U_k and V_k^*, and $\varphi_1, \varphi_2, \dots, \varphi_{m(k)}$ and $\psi_1, \psi_2, \dots, \psi_{m(k)}$ be bases of U_k and V_k^*. Then $u_k = \sum_{i=1}^{m(k)} a_i \varphi_i$, and (2.4) is represented in the form

$$\Big(A(\sum_{i=1}^{m(k)} a_i \varphi_i), \psi_j\Big) = (f, \psi_j), \quad j = 1, 2, \dots, m(k). \tag{2.5}$$

Let P_k^* be an operator of projection of V^* onto V_k^*, and P_k be the adjoint operator of P_k^*, that projects V onto $V_k = V_k^{**}$, where V_k^{**} is the dual space of V_k^*,

$$V_k = \{v | v \in V, \quad (v,g) = 0, \quad g \in \mathcal{N}(P_k^*)\}, \tag{2.6}$$

where $\mathcal{N}(P_k^*)$ is the null-space of P_k^*, see [5], VI, 3.3, VI, 9.18, VI, 9.19, [8], I, 3.4, III, 3.4. There exists infinitely many projections which map V^* onto V_k^*. A projection can be defined as follows:

Let $\alpha_1, \alpha_2, \dots, \alpha_{m(k)}$ be elements of V such that $(\alpha_i, \psi_j) = \delta_{ij}$, where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

Then

$$P_k^* e = \sum_{i=1}^{m(k)} (\alpha_i, e) \psi_i, \quad e \in V^*.$$

In this case $P_k^* = P_k^* P_k^*$.

The projections P_k^* and P_k are uniquely defined, when V is a Hilbert space.

It follows from (2.5) that

$$(Au_k - f, P_k^* h) = 0, \quad h \in V^*.$$

That is, (2.5) is represented in the form

$$P_k A u_k = P_k f. \tag{2.7}$$

Since $V_k^* = (A(U_k))^*$, the assumption $(C2.2)$ implies the existence of a constant $\gamma_k > 0$ such that

$$||P_k A w||_V \geq \gamma_k ||w||_U, \quad w \in U_k, \quad k \in \mathbb{N}. \tag{2.8}$$

Therefore,

$$||(P_k A)^{-1}||_{\mathcal{L}(V_k, U_k)} \leq \frac{1}{\gamma_k}, \quad k \in \mathbb{N}. \tag{2.9}$$

For an arbitrary $y \in U$, we denote the best approximation of y in U_k by $b_k y$. It is defined as follows:

$$||y - b_k y||_U = \inf_{w \in U_k} ||y - w||_U. \tag{2.10}$$

There exists a function $b_k y$ that satisfies (2.10), and such function is unique, when the norm of U is a strictly convex functional.

Theorem 2.1. *Suppose that the conditions* $(C2.1)$, $(C2.2)$, (2.1), *and* (2.2) *are satisfied. Let also* $f \in V$. *Then for any* k *there exists a unique solution* u_k *to the problem* (2.4), *and* $u_k \rightharpoonup u$ *in* U, *where* u *is the solution to the problem* (2.3). *If*

$$\lim_{k \to \infty} \frac{1}{\gamma_k} ||u - b_k u||_U = 0, \tag{2.11}$$

then $u_k \to u$ *in* U. *Furthermore, if there exists* $\gamma > 0$ *such that* $\gamma_k \geq \gamma$ *for all* k, *then*

$$||u_k - u||_U \leq c \inf_{h \in U_k} ||u - h||_U, \tag{2.12}$$

where c *is independent of* k.

Proof. The existence of a unique solution to the problem (2.5) for any k follows from (C2.2) and (2.2). By (C2.2), (2.1), (2.2), and (2.4), we obtain $Au_k \rightharpoonup f$ in V. Therefore, $u_k \rightharpoonup A^{-1} f = u$ in U.

By (2.7) and (2.3), we get

$$P_k A u_k = P_k f = P_k A u. \tag{2.13}$$

(2.8) and (2.13) imply

$$||u_k - b_k u||_U \leq \frac{1}{\gamma_k} ||P_k A(u_k - b_k u)||_V = \frac{1}{\gamma_k} ||P_k A(u - b_k u)||_V \leq \frac{c_1}{\gamma_k} ||u - b_k u||_U. \tag{2.14}$$

Therefore,

$$||u - u_k||_U \leq ||u - b_k u||_U + ||b_k u - u_k||_U \leq (1 + \frac{c_1}{\gamma_k}) ||u - b_k u||_U. \tag{2.15}$$

(2.15) implies that $u_k \to u$ in U, if (2.11) is realized, and if $\gamma_k \geq \gamma > 0$ then (2.12) with $c = 1 + \frac{c_1}{\gamma}$ holds. $\square$

Remark 1. Suppose that the conditions (C2.1), (C2.2), (2.1), and (2.2) are satisfied. Let $f \in V$ and u_k be the solution to the problem (2.4). Let $\hat{U}$ be an arbitrary Banach space such that $U \subset \hat{U}$, and the embedding $U \to \hat{U}$ is compact. Let $\hat{A}$ be an extension of A such that

$\hat{A}$ is an isomorphism of $\hat{U}$ onto $\hat{V}$, and also $V \subset \hat{V}$. Then the function u, that is the solution to the problem (2.3), also is the unique solution to the problem $\hat{A}u = f$, and the Theorem 2.1 implies $u_k \to u$ in $\hat{U}$.

Remark 2 (Second variant). Suppose that the conditions (C2.1) and (C2.2) are satisfied. Let $\check{U}$ and $\check{V}$ be Banach spaces such that $\check{U} \subset U$, the embedding $\check{U} \to U$ is compact. Let also $\check{V} \subset V$, $f \in \check{V}$, and $\check{A}$ be a restriction of A such that $\check{A}$ is an isomorphism of $\check{U}$ onto $\check{V}$. Let $\check{u}$ be the solution to the problem $\check{A}\check{u} = f$. Then $\check{u}$ is also the unique solution to the problem $Au = f$.

Let $\{\check{U}_k\}$, $\{\check{V}_k\}$, and $\{\check{V}_k^*\}$ be sequences of finite-dimensional subspaces of $\check{U}$, $\check{V}$, and $\check{V}^*$ such that

$$\lim_{k\to\infty} \inf_{h\in\check{U}_k} ||v - h||_{\check{U}} = 0, \quad v \in \check{U}, \tag{2.16}$$

$$\check{V}_k = \check{A}(\check{U}_k), \quad \check{V}_k^* = (\check{A}(\check{U}_k))^*. \tag{2.17}$$

Let also $\check{u}_k$ be the solution to the problem

$$\check{u}_k \in \check{U}_k, \quad (\check{A}\check{u}_k, h) = (f, h), \quad h \in \check{V}_k^*. \tag{2.18}$$

Then by Theorem 2.1, we obtain

$$\check{u}_k \rightharpoonup \check{u} \quad \text{in} \quad \check{U}, \quad \check{u}_k \to u \quad \text{in} \quad U. \tag{2.19}$$

Consider the case that $A \in \mathcal{L}(U, U^*)$, i.e. $V = U^*$. Suppose that there exists a constant $\alpha > 0$ such that

$$(Au, u) \geq \alpha ||u||_U^2, \quad u \in U. \tag{2.20}$$

Then by Lax-Milgram theorem, there exists a unique solution u to the problem (2.3) for any $f \in U^*$. The Galerkin approximations u_k of u are defined by (1.2), that is

$$u_k \in U_k, \quad P_k^* A u_k = P_k^* f, \tag{2.21}$$

where P_k^* is the adjoint operator of P_k that projects U onto U_k.

Let $w \in U_k$. Taking (2.20) into account, we obtain

$$||P_k^* Aw||_{U^*} ||w||_U \geq (P_k^* Aw, w) = (Aw, P_k w) = (Aw, w) \geq \alpha ||w||_U^2. \tag{2.22}$$

Therefore,

$$|||P_k^* Aw||_{U^*} \geq \alpha ||w||_U. \tag{2.23}$$

We apply Theorem 2.1. Taking (2.21) into account and remarking that $V = U^*$ and P_k is the projector of U onto U_k, we obtain.

Corollary. Let U be a separable reflexive Banach space. Let $A \in \mathcal{L}(U, U^*)$, $f \in U^*$, and the conditions (2.1), (2.20) be satisfied. Then, for any k, there exists a unique solution u_k to the problem (2.21) and $u_k \to u$ in U, moreover (2.12) is satisfied.

3. **Spaces U_k and V_k^*.**

Let $\{\varphi_i\}_{i=1}^\infty$ be a sequence of elements of U which satisfy the following condition:

(C3.1): $\varphi_1, \varphi_2, \dots, \varphi_k$ are linearly independent for any $k \in \mathbb{N}$, the union of all finite linear combinations $\sum c_i\varphi_i$, $c_i \in \mathbb{R}$, is dense in U.

For any $j \in \mathbb{N}$, we denote the linear span of the elements of the set $\{\varphi_i\}_{i=1}^\infty \setminus \{\varphi_j\}$ by Λ_j. We assume that there exists a constant $\beta > 0$ such that

$$\rho(\varphi_j, \Lambda_j) = \inf_{z\in\Lambda_j} ||\varphi_j - z||_U = \beta_j \geq \beta, \quad j \in \mathbb{N}. \tag{3.1}$$

Consider the case where U is a Hilbert space. By the Schmidt orthogonalization (see e.g. [8], I, 6.3), we can construct a sequence $\{\varphi_i'\}_{i=1}^\infty$ such that, the functions φ_i' are linear combinations of $\varphi_1, \varphi_2, \dots, \varphi_i$, and φ_i' are orthonormal with respect to the scalar product in U. Let Λ_j' be the span of the elements of the set $\{\varphi_i'\}_{i=1}^\infty \setminus \{\varphi_j'\}$.

Then we have

$$\rho(\varphi_j', \Lambda_j') = \inf_{z\in\Lambda_j'} ||\varphi_j' - z||_U = 1, \quad j = \mathbb{N}.$$

Therefore, the functions φ_i' meet the conditions $(C3.1)$ and (3.1) with $\beta_j = 1$.

Theorem 3.1. *Suppose that the conditions $(C2.1)$, $(C2.2)$, $(C3.1)$ and (3.1) are satisfied. Then there exists a sequence $\{\psi_i\}_{i=1}^\infty \subset V^*$ and a positive constant ζ such that*

$$||\psi_i||_{V^*} = 1, \quad (A\varphi_j, \psi_i) = \zeta_i\delta_{ij}, \quad i,j \in \mathbb{N}, \quad \zeta_i = \rho(A\varphi_i, A(\Lambda_i)) \geq \zeta. \tag{3.2}$$

The elements ψ_i are defined uniquely by the conditions

$$||\psi_i||_{V^*} = 1, \quad (A\varphi_j, \psi_i) = 0, \ \text{at} \ \ i \neq j, \tag{3.3}$$

and the linear span of the set $\{\psi_i\}_{i=1}^\infty$ is dense in V^*.

Let U_k and V_k^* be the linear spans of the elements $\varphi_1, \dots, \varphi_k$ and $\psi_1, \dots \psi_k$. Then the condition (2.2) is satisfied and

$$||P_kAw||_V = ||Aw||_V \geq ||A^{-1}||^{-1}_{\mathcal{L}(V,U)}||w||_U, \quad w \in U_k, \quad k \in \mathbb{N}, \tag{3.4}$$

that is

$$||(P_kA)^{-1}||_{\mathcal{L}(V_k,U_k)} = ||A^{-1}||_{\mathcal{L}(V,U)}, \quad k \in \mathbb{N}. \tag{3.5}$$

Proof. By $(C2.2)$ and the Banach theorem on inverse operator, see [22], II, 5, there exists the inverse A^{-1} of the operator A and $A^{-1} \in \mathcal{L}(V,U)$. For an arbitrary $\varepsilon > 0$, there exists $z_\varepsilon \in \Lambda_i$ satisfying

$$\rho(A\varphi_i, A(\Lambda_i)) = ||A\varphi_i - Az_\varepsilon||_V - \varepsilon.$$

Therefore,

$$\rho(A\varphi_i, A(\Lambda_i)) \geq \frac{||\varphi_i - z_\varepsilon||_U}{||A^{-1}||_{\mathcal{L}(V,U)}} - \varepsilon \geq \frac{\rho(\varphi_i, \Lambda_i)}{||A^{-1}||_{\mathcal{L}(V,U)}}.$$

From here and (3.1), we obtain

$$\zeta_i = \rho(A\varphi_i, A(\Lambda_i)) \geq \frac{\beta_i}{||A^{-1}||_{\mathcal{L}(V,U)}} \geq \frac{\beta}{||A^{-1}||_{\mathcal{L}(V,U)}} = \zeta. \tag{3.6}$$

It follows from the known result, see e.g. [7], V, 7, that for any i there exists an element ψ_i which satisfies (3.2). In view of $(C2.2)$ and $(C3.1)$, the elements ψ_i are uniquely defined by

(3.3). On the strength of $(C2.2)$ $(C3.1)$ and (3.2), the spaces V_k^* meet the condition (2.2), and the linear span of the set $\{\psi_i\}_{i=1}^{\infty}$ is dense in V^*.

Let

$$w = \sum_{i=1}^{k} c_i\varphi_i. \tag{3.7}$$

Taking (3.3) into account, we obtain

$$||Aw||_V = \sup_{||\sum_{i=1}^{\infty} b_i\psi_i||_{V^*}=1} \Big(A(\sum_{i=1}^{k} c_i\varphi_i), \sum_{i=1}^{\infty} b_i\psi_i\Big)$$
$$= \sup_{||\sum_{i=1}^{k} b_i\psi_i||_{V^*}=1} \Big(A(\sum_{i=1}^{k} c_i\varphi_i), \sum_{i=1}^{k} b_i\psi_i\Big) = ||P_kAw||_V. \tag{3.8}$$

Therefore, (3.4) and (3.5) are satisfied.

Remark. Suppose that the conditions (C2.1) and (C2.2) are satisfied. Let U_k be a finite-dimensional subspace of U and $\{\varphi_i\}_{i=1}^{m(k)}$ be a basis of U_k. Let $\tilde{P}_k$ be a projection of U onto U_k, and $M_k = (I - \tilde{P}_k)U$, where I is the identical operator in U. Suppose that there exists a sequence $\{\varphi_i\}_{m(k)+1}^{\infty} \subset M_k$ such that the linear span of the set $\{\varphi_i\}_{i=1}^{\infty}$ is dense in U, and (3.1) with $\beta > 0$ is satisfied

Let also $\{\psi_i\}_{i=1}^{m(k)}$ be a set of elements of V^* such that

$$||\psi_i||_{V^*} = 1, \quad (A\varphi_j, \psi_i) = \zeta_i\delta_{ij}, \quad \zeta_i > 0, \quad i, j = 1, 2, ..., m(k). \tag{3.9}$$

Let V_k^* be the linear span of the elements $\psi_1, ..., \psi_{m(k)}$, and P_k^* be a projection of V^* onto V_k^*. Then (3.5), where $P_k = P_k^{**}$ and V_k is the linear span of the elements $A\varphi_1, ..., A\varphi_{m(k)}$, is satisfied.

Consider the case where (3.9) is not satisfied. In this event we have

$$\inf_{||\sum_{i=1}^{m(k)} c_i\varphi_i||_U=1} ||P_k \sum_{i=1}^{m(k)} Ac_i\varphi_i||_V$$
$$= \inf_{||\sum_{i=1}^{m(k)} c_i\varphi_i||_U=1} \sup_{||\sum_{i=1}^{m(k)} b_i\psi_i||_{V^*}=1} \Big(\sum_{i=1}^{m(k)} Ac_i\varphi_i, \sum_{i=1}^{m(k)} b_i\psi_i\Big)$$
$$= \inf_{||\sum_{i=1}^{m(k)} c_i\varphi_i||_U=1} \sup_{||\sum_{i=1}^{m(k)} b_i\psi_i||_{V^*}=1} \sum_{i,j=1}^{m(k)} c_ib_j(A\varphi_i, \psi_j). \tag{3.10}$$

4. **Convergence of the Petrov-Galerkin approximations II.**

We do not assume here that A is a one-to one mapping of U onto V; instead, we suppose

(C4.1): A is a linear continuous mapping of U into V, A has a finite-dimensional kernel N, $N = \{u | u \in U, \ Au = 0\}$,
and the operator A^* adjoint to A, that is defined as
$(Au, v) = (u, A^*v), \ u \in U, \ v \in V^*$,
has a finite-dimensional kernel N^*, $N^* = \{v | v \in V^*, \ A^*v = 0\}$.

(C4.2): The image of A, $ImA = A(U)$, is a closed subspace of V.

Theorem 4.1. *Let U and V be separable reflexive Banach spaces. Let also the conditions $(C4.1)$ and $(C4.2)$ be satisfied. Then the operator A is an isomorphism of U/N onto ImA, and*

$$ImA = \{f | f \in V,\ (f, v) = 0,\ v \in N^*\}. \tag{4.1}$$

Indeed, $(C4.2)$ and the Banach theorem on inverse operator imply that the operator A is an isomorphism of U/N onto ImA. The relation (4.1) follows from $(C4.2)$, see e.g. [7], XII, 2.2.

Let $\check{P}$ be a projection of U onto N. Then we have the decomposition $U = \overset{\circ}{U} \bigoplus N$, where $\overset{\circ}{U} = (I - \check{P})(U)$, I being the identity operator in U.

Let also $\hat{P}$ be a projection of V^* onto N^*. Then $V^* = \overset{\circ}{V}{}^* \bigoplus N^*$, where $\overset{\circ}{V}{}^* = (I_1 - \hat{P})(V^*)$, and I_1 is the identity operator in V^*. (4.1) implies that $\overset{\circ}{V}{}^* = (ImA)^*$, where $(ImA)^*$ is the dual of the space ImA.

Let $\{\overset{\circ}{U}_k\}$ and $\{\overset{\circ}{V}{}^*_k\}$ be sequences of finite dimensional subspaces of $\overset{\circ}{U}$ and $\overset{\circ}{V}{}^*$ such that

$$\lim_{k\to\infty} \inf_{h \in \overset{\circ}{U}_k} ||z - h||_U = 0, \quad z \in \overset{\circ}{U}, \tag{4.2}$$

$$\overset{\circ}{V}{}^*_k = (A(\overset{\circ}{U}_k))^*. \tag{4.3}$$

Consider the problem: Given $f \in ImA$, find u satisfying

$$u \in \overset{\circ}{U}, \quad Au = f. \tag{4.4}$$

By virtue of Theorem 4.1, there exist a unique solution to the problem (4.4) and a positive constant c such that

$$||u||_U \le c||f||_V, \quad f \in ImA. \tag{4.5}$$

The Petrov-Galerkin approximations u_k of u are defined by

$$u_k \in \overset{\circ}{U}_k, \quad (Au_k, h) = (f, h), \quad h \in \overset{\circ}{V}{}^*_k. \tag{4.6}$$

Let P_k^* be an operator of projection of $\overset{\circ}{V}{}^*$ onto $\overset{\circ}{V}{}^*_k$ and P_k be the adjoint of P_k^* operator, that projects ImA onto $\overset{\circ}{V}_k = \overset{\circ}{V}{}^{**}_k = A(\overset{\circ}{U}_k)$,

$$\overset{\circ}{V}_k = \{f | f \in ImA,\ (f, g) = 0,\ g \in \mathcal{N}(P_k^*)\}, \tag{4.7}$$

where $\mathcal{N}(P_k^*)$ is the null-space of P_k^*.

(4.6) is represented in the form

$$P_k A u_k = P_k f. \tag{4.8}$$

Since $\overset{\circ}{V}{}^*_k = (A(U_k))^*$, there exists a constant $\gamma_k > 0$ such that

$$||P_k A w||_V \ge \gamma_k ||w||_U, \quad w \in \overset{\circ}{U}_k, \quad k \in \mathbb{N}. \tag{4.9}$$

By analogy with the proof of Theorem 2.1, we obtain.

Theorem 4.2. *Let U and V be separable reflexive Banach spaces. Suppose that the conditions $(C4.1)$, $(C4.2)$, (4.2) and (4.3) are satisfied. Let also $f = ImA$. Then for any k there exists a* *unique solution to the problem* (4.6) *and* $u_k \rightharpoonup u$ *in* U*, where* u *is the solution to the problem* (4.4)*. If*

$$\lim_{k\to\infty} \frac{1}{\gamma_k} ||u - b_k u||_U = 0, \tag{4.10}$$

where $b_k u$ *is the best approximation of* u*, that is defined by* (2.10) *at* U_k *changed for* $\overset{\circ}{U}_k$*, then* $u_k \to$ *in* U*. If, in addition, there exists* $\gamma > 0$ *such that* $\gamma_k \geq \gamma$ *for all* k*, then*

$$||u_k - u||_U \leq c \inf_{h\in \overset{\circ}{U}_k} ||u - h||_U, \tag{4.11}$$

where c *is independent of* k*.*

The subspaces $\overset{\circ}{U}_k$ of U can be formed as follows: Let $\{U_k\}$ be a sequence of finite-dimensional subspaces of U that satisfy the condition (2.1). Let $\{\varphi_{ik}\}_{i=1}^{G_k}$ be a basis of U_k.

Set

$$\overset{\circ}{\varphi}_{ik} = (I - \check{P})\varphi_{ik}, \quad i = 1, \ldots, G_k. \tag{4.12}$$

We define $\overset{\circ}{U}_k$ as the span of the nonzero elements $\overset{\circ}{\varphi}_{ik}$. Then the spaces $\overset{\circ}{U}_k$ satisfy the condition (4.2).

The nonzero elements $\overset{\circ}{\varphi}_{ik}$ may be linearly dependent. Let M_k be the dimension of $\overset{\circ}{U}_k$. It is obvious that $G_k - n \leq M_k \leq G_k$, where n is the dimension of N.

5. **Problems of the elasticity theory.**

5.1. **Governing equations and a weak solution.** Let Ω be a bounded domain in $\mathbb{R}^2$ occupied by an elastic body. We suppose that the boundary S of Ω consists of two connected components S_1 and S_2, S_1 is the interior boundary, S_2 the exterior boundary, $S_1 \bigcap S_2 = \emptyset$.

The components of the stress tensor $\sigma_{ij}(u)$ are defined by

$$\sigma_{ij}(u) = a_{ijlm}\varepsilon_{lm}(u), \quad i, j, l, m = 1, 2. \tag{5.1}$$

Here $u = (u_1, u_2)$ is the vector-function of displacements, $\varepsilon_{lm}(u)$ are the components of the strain tensor

$$\varepsilon_{lm}(u) = \frac{1}{2}\Big(\frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial x_l}\Big), \tag{5.2}$$

a_{ijlm} are the coefficients of elasticity, depending on $x = (x_1, x_2) \in \Omega$. In (5.1) and below the Einstein convention on summation over repeated index is applied.

The coefficients of elasticity have properties of symmetry

$$a_{ijlm} = a_{jiml} = a_{lmij}, \tag{5.3}$$

and of positive definiteness

$$a_{ijlm}\,\gamma_{ij}\,\gamma_{lm} \geq \alpha\,\gamma_{ij}\,\gamma_{ij}, \quad \alpha = \text{constant } > 0, \quad \gamma_{ij} = \gamma_{ji} \in \mathbb{R}. \tag{5.4}$$

The equations of equilibrium have the form.

$$-\frac{\partial \sigma_{ij}(u)}{\partial x_j} = K_i \text{ in } \Omega, \quad i, j = 1, 2, \tag{5.5}$$

that is

$$-\frac{\partial}{\partial x_j}(a_{1jlm}\varepsilon_{lm}(u)) = K_1 \text{ in } \Omega, \quad j,l,m = 1,2, \tag{5.6}$$

$$-\frac{\partial}{\partial x_j}(a_{2jlm}\varepsilon_{lm}(u)) = K_2 \text{ in } \Omega, \quad j,l,m = 1,2. \tag{5.7}$$

Here K_1 and K_2 are the components of the volume force vector K.

We assign the displacements on S_1 and the surface forces on S_2, i.e.

$$u_i = \hat{u}_i \text{ on } S_1, \quad \sigma_{ij}(u)\nu_j = F_i \text{ on } S_2, \quad i = 1,2, \tag{5.8}$$

where $\hat{u}_i$ and F_i are the components of the prescribed displacements $\hat{u} = (\hat{u}_1, \hat{u}_2)$ and surface forces $F = (F_1, F_2)$, ν_j the components of the unit outward normal $\nu = (\nu_1, \nu_2)$ to S_2.

Let

$$U = \{v | v \in H^1(\Omega)^2, \quad v|_{S_1} = 0\}. \tag{5.9}$$

By virtue of the Korn inequality, the expression

$$||v||_1 = \Big(\int_\Omega \varepsilon_{ij}(v)\varepsilon_{ij}(v)\,dx\Big)^{\frac{1}{2}} \tag{5.10}$$

defines a norm in U that is equivalent to the main norm of $H^1(\Omega)^2$.

We suppose that

$$K = (K_1, K_2) \in U^*, \quad \hat{u} = (\hat{u}_1, \hat{u}_2) \in H^{\frac{1}{2}}(S_1)^2, \quad F = (F_1, F_2) \in H^{-\frac{1}{2}}(S_2)^2, \tag{5.11}$$

$$a_{ijlm} \in L_\infty(\Omega), \quad i,j,l,m = 1,2. \tag{5.12}$$

We also assume that the boundary $S = S_1 \bigcup S_2$ of Ω is Lipschitz continuous. Since $\hat{u} \in H^{\frac{1}{2}}(S_1)^2$, there exists a function w such that

$$w = (w_1, w_2) \in H^1(\Omega)^2, \quad w|_{S_1} = \hat{u}. \tag{5.13}$$

Define the following bilinear form on $U \times U$

$$b(v,h) = \int_\Omega a_{ijlm}\,\varepsilon_{lm}(v)\,\varepsilon_{ij}(h)\,dx, \quad v,h \in U. \tag{5.14}$$

We consider the following problem: Find u^0 satisfying

$$u^0 = (u_1^0, u_2^0) \in U, \tag{5.15}$$

$$b\,(u^0, h) = (g,h), \quad h \in U, \tag{5.16}$$

where

$$(g,h) = (K,h) + (F, h|_{S_2}) - \int_\Omega a_{ijlm}\,\varepsilon_{lm}(w)\,\varepsilon_{ij}(h)\,dx. \tag{5.17}$$

By (5.12), (5.3) and (5.4), the bilinear form b is continuous, symmetric, and positive definite in U. It follows from (5.11), (5.13), and (5.17) that $g \in U^*$. Because of this, the Riesz theorem implies that, there exists a unique solution u^0 to the problem (5.15), (5.16).

Applying the Green formula, it is easy to verify that the function $u = u^0 + w$ is a weak solution to the problem (5.6), (5.7), (5.8).

This solution is unique. If w^1 is another function that satisfies (5.13) and u^1 is the solution to the problem

$$u^1 \in U, \quad b(u^1, h) = (g^1, h), \quad h \in U,$$

where g^1 is defined by the right-hand side of (5.17), in which w is changed for w^1, then $u^0 + w = u^1 + w^1$.

Therefore, the following assertion is valid:

Theorem 5.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitz continuous boundary $S = S_1 \bigcup S_2$, where S_1 and S_2 are connected components of S. Suppose that the conditions* (5.3), (5.4), (5.11), *and* (5.12) *are satisfied. Then there exists a unique weak solution u to the problem* (5.6), (5.7), (5.8). *The function u is presented in the form $u = w + u^0$, where w satisfies the condition* (5.13) *and u^0 is the solution to the problem* (5.15), (5.16). *There exists $c > 0$ such that*

$$||u||_{H^1(\Omega)^2} \leq c\Big(||K||_{U^*} + ||\hat{u}||_{H^{\frac{1}{2}}(S_1)^2} + ||F||_{H^{-\frac{1}{2}}(S_2)^2}\Big). \tag{5.18}$$

Restrictions on stiffness and strength are imposed on solutions of the problems of the theory of elasticity, see [12], [21]. These restrictions may be taken in the form

$$\begin{aligned}
&\max_{x\in\overline{\Omega}} |u(x)| \leq c_1, \\
&\max_{x\in\overline{\Omega}}((\Pi_{ij}\sigma_{ij}(u))(x) + (\Pi_{ijlm}\sigma_{ij}(u)\sigma_{lm}(u))(x)) \leq 1,
\end{aligned} \tag{5.19}$$

where Π_{ij} and Π_{ijlm} are the components of the strength tensors.

It follows from here that the solution u to the problem (5.6), (5.7), (5.8) should be of the class $C^1(\overline{\Omega})^2$. Because of this, we will now deal with such solution to this problem.

5.2. **Regular solutions and the solution to the conjugate problem.** We suppose that

$$a_{ijlm} \in C^1(\overline{\Omega}), \quad i,j,l,m = 1,2, \tag{5.20}$$

$$K = (K_1, K_2) \in L_p(\Omega)^2, \quad \hat{u} = (\hat{u}_1, \hat{u}_2) \in W_p^{2-\frac{1}{p}}(S_1)^2, \quad F = (F_1, F_2) \in W_p^{1-\frac{1}{p}}(S_2)^2, \quad p > 1, \tag{5.21}$$

(C5.1): Ω is a bounded domain in $\mathbb{R}^2$, the boundary S of Ω consists of two connected components S_1 and S_2, such that $S_1 \bigcap S_2 = \emptyset$, S_1 and S_2 are of the class C^2.

Define an operator A as follows:

$$A = (A_1, A_2), \quad A_1 = (A_{11}, A_{12}), \quad A_2 = (A_{21}, A_{22}), \quad u = (u_1, u_2), \tag{5.22}$$

$$A_{11}u = -\frac{\partial}{\partial x_j}(a_{1jlm}\varepsilon_{lm}(u)), \quad A_{12}u = -\frac{\partial}{\partial x_j}(a_{2jlm}\varepsilon_{lm}(u)) \text{ in } \Omega, \tag{5.23}$$

$$A_{21}u = u \text{ on } S_1, \quad A_{22}u = \left\{\begin{array}{c} \sigma_{1j}(u)\nu_j \\ \sigma_{2j}(u)\nu_j \end{array}\right\} \text{ on } S_2. \tag{5.24}$$

We consider the following problem: Find u satisfying

$$\begin{gathered}
u = W_p^2(\Omega)^2, \\
A_1 u = \left\{\begin{array}{c} A_{11}u \\ A_{12}u \end{array}\right\} = K \text{ in } \Omega, \quad A_{21}(u) = \hat{u} \text{ on } S_1, \quad A_{22}(u) = F \text{ on } S_2.
\end{gathered} \tag{5.25}$$

Theorem 5.2. *Suppose that the conditions* (5.3), (5.4), (5.20), (5.21), *and* (C5.1) *are satisfied. Then there exists a unique solution to the problem* (5.25), *and*

$$||u||_{W_p^2(\Omega)^2} \leq c\Big(||K||_{L_p(\Omega)^2} + ||\hat{u}||_{W_p^{2-\frac{1}{p}}(S_1)^2} + ||F||_{W_p^{1-\frac{1}{p}}(S_2)^2}\Big). \tag{5.26}$$

The operator $A=\{A_1=(A_{11},A_{12}),\ A_2=(A_{21},A_{22})\}$ *is an isomorphism of* $W^2_p(\Omega)^2$ *onto*

$$V=L_p(\Omega)^2\times W_p^{2-\frac{1}{p}}(S_1)^2\times W_p^{1-\frac{1}{p}}(S_2)^2.$$

Remark. Since $W^2_p(\Omega)\subset C^1(\overline{\Omega})$ at $n=2$ and $p>2$, the components of the stress tensor $\sigma_{ij}(u)$ belong to the space $C(\overline{\Omega})$, when $u\in W^2_p(\Omega)^2$ with $p>2$. In this case the restrictions (5.19) have sense.

Proof of Theorem 5.2. By (5.3) the operator A_1 is represented in the form

$$A_{11}u=-\frac{\partial}{\partial x_j}\Big(a_{1jlm}\frac{\partial u_l}{\partial x_m}\Big),\quad A_{12}u=-\frac{\partial}{\partial x_j}\Big(a_{2jlm}\frac{\partial u_l}{\partial x_m}\Big). \tag{5.27}$$

The matrix $b(x,\xi)$ with elements

$$b_{il}(x,\xi)=a_{ijlm}(x)\xi_j\xi_m \tag{5.28}$$

is associated with the principal part of the operator $-A_1$ and the vector $\xi=(\xi_1,\xi_2)$, see [1], [17], [19]. In this case

$$b_{il}(x,\xi)\eta_l=a_{ijlm}(x)\varepsilon_j\eta_l\xi_m,\quad b_{il}(x,\xi)\eta_l\eta_i=a_{ijlm}(x)\eta_i\xi_j\eta_l\xi_m,\quad \eta=(\eta_1,\eta_2)\in\mathbb{R}^2. \tag{5.29}$$

Following the Lemma 3.2 from [15], let us show that

$$L(x,\xi)=\det||a_{ijlm}(x)\xi_j\xi_m||\neq 0\ \text{ for }\xi\neq 0,\ \ x\in\overline{\Omega}. \tag{5.30}$$

Taking (5.3) and (5.4) into account, we obtain

$$a_{ijlm}(x)\eta_i\,\xi_j\,\eta_l\,\xi_m=\frac{1}{4}\,a_{ijlm}(x)(\eta_i\xi_j+\eta_j\xi_i)(\eta_l\xi_m+\eta_m\xi_l)\geq\frac{1}{4}\,\alpha\sum_{i.j=1}^{2}(\eta_i\xi_j+\eta_j\xi_i)^2. \tag{5.31}$$

If $a_{ijlm}(x)\eta_i\,\xi_j\,\eta_l\,\xi_m=0$, then

$$\eta_i\xi_j+\eta_j\xi_i=0,\quad i,j=1,2.$$

We suppose that $\xi\neq 0$. Multiplying each of these equations by $\eta_i\xi_j$ and summing over i and j , we obtain $|\eta|^2|\xi|^2+|\eta_i\xi_i|^2=0$; that is $\eta=0$. Thus, for any $\xi\neq 0$ and $x\in\overline{\Omega}$, $a_{ijlm}(x)\eta_i\,\xi_j\,\eta_l\,\xi_m>0$ for $\eta\neq 0$, and (5.30) holds.

It follows from (5.30) and (5.20) that there exists a positive constant μ such that

$$\mu^{-1}|\xi|^4\leq|L(x,\xi)|\leq\mu|\xi|^4,\quad x\in\overline{\Omega},\quad \xi\in\mathbb{R}^2. \tag{5.32}$$

Consequently, the operator A_1 is uniformly elliptic.

$L(x,\xi)$ is a polynomial of fourth degree with respect to ξ with real coefficients. Because of this, for every pair of linearly independent vectors $\xi=(\xi_1,\xi_2)$ and $\xi^1=(\xi^1_1,\xi^1_2)$ from $\mathbb{R}^2$, the polynomial $L(x,\xi+\tau\xi^1)$ in the complex variable τ has two roots with positive imaginary part. Therefore, the supplementary condition on L is satisfied.

The complementing boundary condition is also satisfied, because there is no a constant c such that (5.18) holds, if the complementing boundary condition is violated, see [1].

Now it follows from [1, 19] that if a solution u of the problem (5.25) belongs to $W^2_p(\Omega)^2$, then there exists $\tilde{c}>0$ such that

$$||u||_{W^2_p(\Omega)^2}\leq\tilde{c}\Big(||K||_{L_p(\Omega)^2}+||\hat{u}||_{W_p^{2-\frac{1}{p}}(S_1)^2}+||F||_{W_p^{1-\frac{1}{p}}(S_2)^2}+||u||_{L_p(\Omega)^2}\Big).$$

From here and the Peetre lemma, see [10], Chapter 2, Section 5.2, [16], it follows that the image of the operator A is a closed subspace in V and the kernel of A is a finite-dimensional subspace of $W^2_p(\Omega)$.

Denote the kernels of the operators A and the adjoint of it A^* by N and N^*, respectively. It follows from Theorem 5.1 that the problem $Au = 0$ has zero solution only. Therefore, $N = \{0\}$.

Let v and h be two smooth vector-functions given in $\overline{\Omega}$. By applying the Green formula, we obtain

$$\begin{aligned}\int_\Omega a_{ijlm}\,\varepsilon_{lm}(v)\,\varepsilon_{ij}(h)\,dx &= -\int_\Omega \Big(\frac{\partial}{\partial x_j}(a_{ijlm}\,\varepsilon_{lm}(v))\Big)\,h_i\,dx \\ +\int_S a_{ijlm}\,\varepsilon_{lm}(v)\nu_j h_i\,ds &= -\int_\Omega \Big(\frac{\partial}{\partial x_j}(a_{ijlm}\,\varepsilon_{lm}(h))\Big)\,v_i\,dx \\ &\quad+\int_S a_{ijlm}\,\varepsilon_{lm}(h)\nu_j v_i\,ds. \end{aligned} \tag{5.33}$$

Taking into account that $S = S_1 \bigcup S_2$, we obtain from (5.33) that

$$\begin{aligned}&\int_\Omega \Big(\frac{\partial}{\partial x_j}(a_{ijlm}\,\varepsilon_{lm}(v))\Big)\,h_i\,dx + \int_{S_1} v_i a_{ijlm}\,\varepsilon_{lm}(h)\nu_j\,ds - \int_{S_2} a_{ijlm}\,\varepsilon_{lm}(v)\nu_j h_i\,ds \\ &= \int_\Omega \Big(\frac{\partial}{\partial x_j}(a_{ijlm}\,\varepsilon_{lm}(h))\Big)\,v_i\,dx + \int_{S_1} h_i a_{ijlm}\,\varepsilon_{lm}(v)\nu_j\,ds - \int_{S_2} a_{ijlm}\,\varepsilon_{lm}(h)\nu_j v_i\,ds.\end{aligned} \tag{5.34}$$

Therefore, the operator A is formally selfadjoint, and hence $N^* = \{0\}$.

Thus, A is a one-to-one mapping of $W_p^2(\Omega)^2$ onto V, and (5.26) follows from the Banach theorem on inverse operator. Hence, A is an isomorphism of $W_p^2(\Omega)$ onto V. □

Consider the problem (5.25) provided that

$$a_{ijlm} \in C^{1,1}(\overline{\Omega}), \quad i,j,l,m = 1,2, \tag{5.35}$$

$$\begin{aligned}K = (K_1, K_2) \in W_p^\alpha(\Omega)^2, \quad \hat{u} = (\hat{u}_1, \hat{u}_2) \in W_p^{2+\alpha-\frac{1}{p}}(S_1)^2, \\ F = (F_1, F_2) \in W_p^{1+\alpha-\frac{1}{p}}(S_2)^2, \quad \alpha \in (0,1),\ p > 1.\end{aligned} \tag{5.36}$$

The next result follows from the proof of Theorem 5.2 and Theorem 10.1.1 in [17].

Theorem 5.3. *Suppose that the conditions* (5.3), (5.4), (5.35), *and* (5.36) *are satisfied. Let also the connected components S_1 and S_2 of the boundary S be of the class $S^{2,1}$. Then there exists a unique $u \in W_p^{2+\alpha}(\Omega)^2$ that is the solution to the problem* (5.25).

Solving problems on optimization and optimal control systems, depicted by partial and ordinary differential equations, is directly associated with solving problems for the operator, that is adjoint to the one that governs the process under consideration, see [11, 12].

Let us consider the boundary value problem for the operator A^*, that is adjoint to A defined by (5.22)–(5.24).

We have

$$(Ah, v) = (h, A^* v), \quad h \in W_p^2(\Omega)^2, \quad v \in V^*, \quad A^* \in \mathcal{L}(V^*, (W_p^2(\Omega)^2)^*), \tag{5.37}$$

where

$$V^* = L_q(\Omega)^2 \times W_q^{-2+\frac{1}{p}}(S_1)^2 \times W_q^{-1+\frac{1}{p}}(S_2)^2, \quad q = \frac{p}{p-1}, \quad p > 1. \tag{5.38}$$

Theorem 5.4. *Suppose that the operator A is defined by* (5.22), (5.23), (5.24), *and the conditions* (5.3), (5.4), (5.20), *and* ($C5.1$) *are satisfied. Then for an arbitrary $g \in (W_p^2(\Omega)^2)^*$, there exists a unique $v \in V^*$ such that $A^* v = g$, that is*

$$(Ah, v) = (g, h), \quad h \in W_p^2(\Omega)^2. \tag{5.39}$$

The operator A^* is an isomorphism of V^* onto $(W_p^2(\Omega)^2)^*$.

Indeed, since the operator A is an isomorphism of $W_p^2(\Omega)^2$ onto V, there exists the inverse operator A^{*-1} of A^*, and $A^{*-1} \in \mathcal{L}((W_p^2(\Omega)^2)^*, V^*)$, $A^{*-1} = A^{-1*}$, see e.g. [8], Chapter III, Theorem 5.30. Therefore, for any $g \in (W_p^2(\Omega)^2)^*$, there exists a unique $v \in V^*$ such that $A^* v = g$, and the operator A^* is an isomorphism of V^* onto $(W_p^2(\Omega)^2)^*$. The equality (5.39) follows from (5.37).

5.3. **Problem with forces on S_1 and S_2.**

5.3.1. **Weak solution.** Let us consider the case where the boundary operator $A_2 = (A_{21}, A_{22})$ has the following form:

$$A_{21}\, u = \{a_{ijlm}\, \varepsilon_{lm}(u)\nu_j\}_{i=1}^2 \text{ on } S_1, \quad A_{22}\, u = \{a_{ijlm}\, \varepsilon_{lm}(u)\nu_j\}_{i=1}^2 \text{ on } S_2. \tag{5.40}$$

We deal with the problem: Find u satisfying,

$$A_1\, u = \begin{Bmatrix} A_{11}\, u \\ A_{12}\, u \end{Bmatrix} = K \text{ in } \Omega, \quad A_2\, u = \begin{Bmatrix} A_{21}\, u \\ A_{22}\, u \end{Bmatrix} = \begin{Bmatrix} F^1 \\ F^2 \end{Bmatrix}. \tag{5.41}$$

Here the operator A_1 is defined by (5.23), K, F^1, and F^2 are given functions of volume and surface forces.

Define an operator $\overset{\circ}{A} \in \mathcal{L}(H^1(\Omega)^2, (H^1(\Omega)^2)^*)$ by the relation

$$(\overset{\circ}{A}v, h) = \int_\Omega a_{ijlm}\, \varepsilon_{lm}(v)\, \varepsilon_{ij}(h)\, dx, \quad v, h \in H^1(\Omega)^2. \tag{5.42}$$

The kernel N of the operator $\overset{\circ}{A}$ consist of all functions v such that

$$v \in H^1(\Omega)^2, \quad \int_\Omega a_{ijlm}\, \varepsilon_{lm}(v)\, \varepsilon_{ij}(h)\, dx = 0, \quad h \in H^1(\Omega)^2. \tag{5.43}$$

By (5.4) we obtain from (5.43) that $\varepsilon_{lm}(v) = 0$, $l, m = 1, 2$. Therefore, N is the space of the rigid displacements, which has the following form:

$$N = \{v | v = (v_1, v_2),\ v_1 = a_1 + a_3 x_2,\ v_2 = a_2 - a_3 x_1,\ a_1, a_2, a_3 \in \mathbb{R}\}. \tag{5.44}$$

It follows from here that the functions

$$w_1 = (1, 0), \quad w_2 = (0, 1), \quad w_3 = (x_2, -x_1) \tag{5.45}$$

form a basis of N.

The space $H^1(\Omega)^2$ is represented in the form

$$H^1(\Omega)^2 = \mathcal{U} \bigoplus N. \tag{5.46}$$

Here the subspace $\mathcal{U}$ can be defined so that the spaces $\mathcal{U}$ and N are mutually orthogonal with respect to the scalar product of $H^1(\Omega)^2$ or $L_2(\Omega)^2$. We consider that $\mathcal{U}$ and N are mutually orthogonal with respect to the scalar product of $L_2(\Omega)^2$. It follows from (5.45) and the Korn inequality, see [12], 1.7, that the expression

$$||v||_1 = (\overset{\circ}{A}v, v)^{\frac{1}{2}} + \left(\left(\int_\Omega v_1\, dx \right)^2 + \left(\int_\Omega v_2\, dx \right)^2 + \left(\int_\Omega (x_2 v_1 - x_1 v_2)\, dx \right)^2 \right)^{\frac{1}{2}} \tag{5.47}$$

defines a norm in $H^1(\Omega)^2$ that is equivalent to the main norm of $H^1(\Omega)^2$. The first, second, and third integrals in (5.47) being the projections of v onto the basis functions w_1, w_2, w_3 with respect to the scalar product of $L_2(\Omega)^2$. Therefore, the expressions $(\overset{\circ}{A}v, h)$ and $(\overset{\circ}{A}v, v)^{\frac{1}{2}}$ define a scalar product and a norm in the factor space $H^1(\Omega)^2/N$ and in $\mathcal{U}$.

We suppose that

$$K = (K_1, K_2) \in (H^1(\Omega)^2)^*,\ F^1 = (F_1^1, F_2^1) \in H^{-\frac{1}{2}}(S_1)^2,$$
$$F^2 = (F_1^2, F_2^2) \in H^{-\frac{1}{2}}(S_2)^2, \tag{5.48}$$
$$(K, w_i) + (F^1, w_i|_{S_1}) + (F^2, w_i|_{S_2}) = 0, \quad i = 1, 2, 3. \tag{5.49}$$

By (5.48) and (5.49), the functional J defined as

$$(J, h) = (K, h) + (F^1, h|_{S_1}) + (F^2, h|_{S_2}), \quad h \in H^1(\Omega)^2,$$

belong to the space $(H^1(\Omega)^2/N)^*$.

Consider the problem: Find u satisfying:

$$u \in \mathcal{U}; \quad (\overset{\circ}{A}u, h) = (K, h) + (F^1, h|_{S_1}) + (F^2, h|_{S_2}), \quad h \in \mathcal{U}. \tag{5.50}$$

Bearing in mind (5.33), we can see, that if u is a solution to the problem (5.50), then u is a weak solution to the problem (5.41).

By using the Riesz theorem, we obtain the following result:

Theorem 5.5. *Suppose that the conditions* (5.3), (5.4), (5.12), (5.48), *and* (5.49) *are satisfied. Then there exists a unique solution u to the problem* (5.50), *and*

$$||u||_{H^1(\Omega)^2} \le c \left(||K||_{(H^1(\Omega)^2)^*} + ||F^1||_{H^{-\frac{1}{2}}(S_1)^2} + ||F^2||_{H^{-\frac{1}{2}}(S_2)^2} \right), \tag{5.51}$$

the operator $\overset{\circ}{A}$ is an isomorphism of $\mathcal{U}$ onto $\mathcal{U}^*$.

5.3.2. **Regular solutions.** At the conditions (5.20) and (C5.1), the operator $A = (A_1, A_2)$, defined by (5.22), (5.23), and (5.40), is a linear continuous mapping of $W_p^2(\Omega)^2$ into

$$V = L_p(\Omega)^2 \times W_p^{1-\frac{1}{p}}(S_1)^2 \times W_p^{1-\frac{1}{p}}(S_2)^2, \quad p > 1. \tag{5.52}$$

By (5.33) the operator A is formally self-adjoint, the kernel N of A is defined by (5.44), and the kernel N^* of the operator A^* has the form

$$N^* = \{\mathcal{R} | \mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3),\ \mathcal{R}_1 = (a_1 + a_3 x_2, a_2 - a_3 x_1) \text{ in } \Omega,$$
$$\mathcal{R}_2 = (a_1 + a_3 x_2, a_2 - a_3 x_1) \text{ on } S_1, \mathcal{R}_3 = (a_1 + a_3 x_2, a_2 - a_3 x_1) \text{ on } S_2,\ a_1, a_2, a_3 \in \mathbb{R}\}. \tag{5.53}$$

Define the following set:

$$\overset{\circ}{V} = \{(K, F^1, F^2) | (K, F^1, F^2) \in V, \int_\Omega K_i\, dx + \int_{S_1} F_i^1\, ds + \int_{S_2} F_i^2\, ds = 0,\ i = 1, 2,$$
$$\int_\Omega (K_1 x_2 - K_2 x_1)\, dx + \int_{S_1} (F_1^1 x_2 - F_2^1 x_1)\, ds + \int_{S_2} (F_1^2 x_2 - F_2^2 x_1)\, ds = 0\}. \tag{5.54}$$

We consider the functions w_1,w_2,w_3, which are defined by (5.45), as elements of $L_2(\Omega)^2$.

Let

$$\tilde{w}_1 = \left(\left(\int_\Omega dx \right)^{-\frac{1}{2}}, 0 \right),\ \tilde{w}_2 = \left(0, \left(\int_\Omega dx \right)^{-\frac{1}{2}} \right),$$
$$\tilde{w}_3 = \frac{w_3 - \sum_{i=1}^2 (w_3, \tilde{w}_i)\tilde{w}_i}{||w_3 - \sum_{i=1}^2 (w_3, \tilde{w}_i)\tilde{w}_i||_{L_2(\Omega)^2}}. \tag{5.55}$$

The family $\{\tilde{w}_i\}_{i=1}^3$ is a basis of N, that is orthonormal with respect to the scalar product in $L_2(\Omega)^2$.

We denote the projection of $L_2(\Omega)^2$ onto N by $\check{P}$

$$v \in L_2(\Omega)^2, \quad \check{P}v = \sum_{i=1}^3 (v, \tilde{w}_i)\tilde{w}_i. \tag{5.56}$$

Since $W_p^2(\Omega)^2 \subset L_2(\Omega)^2$, we have

$$W_p^2(\Omega)^2 = \overset{\circ}{U} \bigoplus N, \quad \overset{\circ}{U} = (I - \check{P})\{W_p^2(\Omega)^2\}, \tag{5.57}$$

I being the identity operator.

Theorem 5.6. *Suppose that the conditions* $(C5.1)$, (5.3), (5.4), *and* (5.20) *are satisfied. Then for an arbitrary* $(K, F^1, F^2) \in \overset{\circ}{V}$, *there exists a unique* $u \in \overset{\circ}{U}$ *that is the solution to the problem* (5.41), *and*

$$||u||_{W_p^2(\Omega)^2} \le c \left(||K||_{L_p(\Omega)^2} + ||F^1||_{W_p^{1-\frac{1}{p}}(S_1)^2} + ||F^2||_{W_p^{1-\frac{1}{p}}(S_2)^2} \right), \tag{5.58}$$

the operator $A = (A_1, A_2)$ *is an isomorphism of* $W_p^2(\Omega)^2/N$ *onto* $\overset{\circ}{V}$.

The proof of this theorem is closely similar to the proof of the Theorem 5.2. Because of this, we present a sketch of the proof.

It follows from the proof of the Theorem 5.2 that the operator A_1 is uniformly elliptic and the supplementary condition on L is satisfied. By (5.51) the complementing condition is fulfilled.

Now it follows from [1], [19], that if a solution to the problem (5.41) u belongs to $W_p^2(\Omega)^2$, then the following inequality holds:

$$||u||_{W_p^2(\Omega)^2} \le c_1 \left(||K||_{L_p(\Omega)^2} + ||F^1||_{W_p^{1-\frac{1}{p}}(S_1)^2} + ||F^2||_{W_p^{1-\frac{1}{p}}(S_2)^2} + ||u||_{L_p(\Omega)^2} \right). \tag{5.59}$$

(5.59) and the Peetre Lemma imply that the image of the operator A is a closed subspace in V. Therefore, there exists a solution to the problem (5.41) if $(K, F^1, F^2) \in V$ and

$$(K, h) + (F^1, h|_{S_1}) + (F^2, h|_{S_2}) = 0, \quad h \in N^*,$$

see e.g. [7], XII, 2. These conditions exactly mean that $(K, F^1, F^2) \in \overset{\circ}{V}$. Thus, for any $(K, F^1, F^2) \in \overset{\circ}{V}$, there exists a unique $u \in \overset{\circ}{U}$, that is a solution to the problem (5.41) and (5.58) is satisfied. The operator A is an isomorphism of $W_p^2(\Omega)^2/N$ onto $\overset{\circ}{V}$.

The next theorem follows from the proof of Theorem 5.6 and Theorem 10.1.1 in [17].

Theorem 5.7. *Suppose that the conditions* (5.3), (5.4), *and* (5.35) *are satisfied, and the components S_1 and S_2 of the boundary S are of the class $C^{2,1}$ Let also*

$$K = (K_1, K_2) \in W_p^{\alpha}(\Omega)^2, \quad F^1 = (F_1^1, F_2^1) \in W_p^{1+\alpha-\frac{1}{p}}(S_1)^2,$$
$$F^2 = (F_1^2, F_2^2) \in W_p^{1+\alpha-\frac{1}{p}}(S_2)^2, \tag{5.60}$$

and (5.54) *holds. Then there exists a unique $u \in W_p^{2+\alpha}(\Omega)/N$ that is the solution to the problem* (5.41).

5.4. **Three-dimensional problems.** We consider two problems of the elasticity theory in a three-dimensional domain Ω. We suppose

(C5.2): Ω is a bounded domain in $\mathbb{R}^3$ with a boundary S of the class C^2.

Let $u = (u_1, u_2, u_3)$ be a vector-function of displacements. The components of the stress tensor σ_{ij} are defined by (5.1), where $i, j, l, m = 1, 2, 3$, and $\varepsilon_{lm}(u)$ are determined by (5.2). Define an operator A as follows:

$$A = (A_1, A_2),$$
$$A_1 = (A_{11}, A_{12}, A_{13}), \quad A_{1i}u = \frac{\partial}{\partial x_j}(a_{ijlm}\varepsilon_{lm}(u)) \text{ in } \Omega, \quad i = 1, 2, 3, \tag{5.61}$$
$$A_2 = (A_{21}, A_{22}, A_{23}), \quad A_{2i}u = u_i \text{ on } S, \quad i = 1, 2, 3. \tag{5.62}$$

We consider the following problem: Find u satisfying

$$u \in W_p^2(\Omega)^3, \quad A_1 u = K \text{ in } \Omega, \quad A_2 u = \hat{u} \text{ on } S. \tag{5.63}$$

Here, we assume that

$$K = (K_1, K_2, K_3) \in L_p(\Omega)^3, \quad \hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3) \in W_p^{2-\frac{1}{p}}(S)^3, \quad p > 1. \tag{5.64}$$

Theorem 5.8. *Suppose that the conditions* (5.3), (5.4), (5.64), *and* (5.20), *where $i, j, l, m = 1, 2, 3$, are satisfied. Let also* $(C5.2)$ *is valid. Then there exists a unique solution to the problem* (5.63), *and there exists $c > 0$ such that*

$$||u||_{W_p^2(\Omega)^3} \le c\left(||K||_{L_p(\Omega)^3} + ||\hat{u}||_{W_p^{2-\frac{1}{p}}(S)^3}\right). \tag{5.65}$$

The operator A is an isomorphism of $W_p^2(\Omega)^3$ onto $V = L_p(\Omega)^3 \times W_p^{2-\frac{1}{p}}(S)^3$.

The proof of this theorem is closely similar to the proof of Theorem 5.2. Because of this, we will dwell on some steps of the proof.

By repeating the arguments of the proof of Theorem 5.2, we obtain that the operator A_1 is elliptic and the supplementary condition is satisfied. The Dirichlet boundary condition (5.62) is complementing, see [1].

It follows from here that the image of the operator A is a closed subspace of V and the kernel of A is a finite-dimensional subspace of $W_p^2(\Omega)^3$.

Let $v \in W_p^2(\Omega)^3$ and $Av = 0$, that is $A_1 v = 0$, $v = 0$, on S.

From here and (5.4), we obtain

$$0 = -\int_{\Omega} \Big(\frac{\partial}{\partial x_j}(a_{ijlm}\varepsilon_{lm}(v))\Big) v_i \, dx = \int_{\Omega} (a_{ijlm}\varepsilon_{lm}(v)\varepsilon_{ij}(v) \, dx \geq \alpha \int_{\Omega} \sum_{i,j=1}^{3} (\varepsilon_{ij}(v))^2 \, dx. \tag{5.66}$$

The Korn inequality and (5.66) yield $v = 0$, that is $N = \{0\}$, where N is the kernel of A.

It follows from (5.33) that

$$\begin{aligned} &\int_{\Omega} \Big(\frac{\partial}{\partial x_j}(a_{ijlm}\varepsilon_{lm}(v))\Big) h_i \, dx + \int_S v_i a_{ijlm}\varepsilon_{lm}(h)\nu_j \, ds \\ &= \int_{\Omega} \Big(\frac{\partial}{\partial x_j}(a_{ijlm}\varepsilon_{lm}(h))\Big) v_i \, dx + \int_S h_i a_{ijlm}\varepsilon_{lm}(v)\nu_j \, ds. \end{aligned} \tag{5.67}$$

Therefore, the operator A is formally self-adjoint, and the kernel of the adjoint operator A^* contains zero element of V^* only. Thus, A is an isomorphism of $W_p^2(\Omega)^3$ onto $L_p(\Omega)^3 \times W_p^{2-\frac{1}{p}}(S)^3$, and (5.65) holds.

Consider the problem, where surface forces are prescribed on S. In this case, the boundary operator A_2 is defined by

$$A_2 = (A_{21}, A_{22}A_{23}), \quad A_{2i}u = a_{ijlm}\varepsilon_{lm}(u)\nu_j \text{ on } S, \quad i = 1,2,3, \tag{5.68}$$

where ν_j are components of the unit outward normal $\nu = (\nu_1, \nu_2, \nu_3)$ to S.

We consider the problem: Find u such that

$$u \in W_p^2(\Omega)^3, \quad A_1 u = K, \quad A_2 u = F, \tag{5.69}$$

where A_1 is defined by (5.61).

Taking (5.4) into account, we obtain from (5.33), that the kernel N of the operator $A = (A_1, A_2)$ consists of functions v such that $\varepsilon_{ij}(v) = 0$, $i, j = 1,2,3$. Therefore,

$$N = \{v | v = (v_1, v_2, v_3), \ v_i = a_i + b_{ik}x_k, \ a_i, b_{ik} \in \mathbb{R}, \ b_{ik} = -b_{ki}, \ i,k = 1,2,3\}. \tag{5.70}$$

It follows from (5.33) that the kernel N^* of the operator A^* is defined as follows:

$$\begin{aligned} &N^* = \{\mathcal{B} | \mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2), \ \mathcal{B}_j = \{\mathcal{B}_{ji}\}_{i=1}^3, \ j = 1,2, \\ &\mathcal{B}_{1i} = a_i + b_{ik}x_k \text{ in } \Omega, \ \mathcal{B}_{2i} = a_i + b_{ik}x_k \text{ on } S, \ a_i, b_{ik} \in \mathbb{R}, \ b_{ik} = -b_{ki}, \ i,k = 1,2,3\}. \end{aligned} \tag{5.71}$$

The following functions z_i form a basis of N:

$$\begin{aligned} &z_1 = (1,0,0), \quad z_2 = (0,1,0), \quad z_3 = (0,0,1), \\ &z_4 = (x_2, -x_1, 0), \quad z_5 = (x_3, 0, -x_1), \quad z_6 = (0, x_3, -x_2), \end{aligned} \tag{5.72}$$

and the functions y_i, defined as

$$y_i = (y_{i1}, y_{i2}), \quad y_{i1} = z_i \text{ in } \Omega, \quad y_{i2} = z_i \text{ on } S, \quad i = 1,2,\dots,6, \tag{5.73}$$

form a basis of N^*.

We set

$$W = \{(K,F) | (K,F) \in L_p(\Omega)^3 \times W_p^{1-\frac{1}{p}}(S)^3 = V, \ p > 1, \ (K,h) + (F, h|_S) = 0, \ h \in N^*\}. \tag{5.74}$$

The condition $(K, h) + (F, h|_S) = 0$, $h \in N^*$ is equivalent to the following equations:

$$
\begin{aligned}
&\int_\Omega K_i\,dx + \int_S F_i\,ds = 0, \quad i = 1,2,3,\\
&\int_\Omega (K_1x_2 - K_2x_1)\,dx + \int_S (F_1x_2 - F_2x_1)\,ds = 0,\\
&\int_\Omega (K_1x_3 - K_3x_1)\,dx + \int_S (F_1x_3 - F_3x_1)\,ds = 0,\\
&\int_\Omega (K_2x_3 - K_3x_2)\,dx + \int_S (F_2x_3 - F_3x_2)\,ds = 0.
\end{aligned} \tag{5.75}
$$

By analogy with the proof of Theorem 5.5, we obtain

Theorem 5.9. *Let the operator* $A = (A_1, A_2)$ *be defined by* (5.61), (5.68). *Suppose that the conditions* (5.3), (5.4), *and* (5.20), *where* $i, j, l, m = 1, 2, 3$, *are satisfied. Assume also that* $(C5.2)$ *holds and* $(K, F) \in W$. *Then there exists a solution to the problem* (5.69), *and the operator* A *is an isomorphism of* $W_p^2(\Omega)^3/N$ *onto* W.

6. The Petrov-Galerkin method for the problems of the elasticity theory.

6.1. **Spaces $\mathbf{U_k}$.** Let $[a, b]$ be a segment in $\mathbb{R}$ and Δ be a partition of it

$$\Delta : a = y_0 < y_1 < \cdots < y_k = b.$$

We denote by $S_{3i}(\Delta, [a, b])$ the space of cubic splines consisting of functions w such that $w \in C^i([a, b])$, $i = 1$ or 2, and on each subsegment $[y_j, y_{j+1}]$, the function w is a polynomial of the degree 3. The space $S_{31}(\Delta, [a, b])$ is said to be the Hermite cubic splines.

Bases of the spaces $S_{3i}(\Delta, [a, b])$ are formed by corresponding cardinal splines, whose supports have the minimal length, see [2, 23].

Let Ω_1 be a rectangular domain such that

$$\Omega_1 = \{x | x = (x_1, x_2),\ a_1 < x_1 < b_1,\ a_2 < x_2 < b_2\}, \quad \Omega \subset \Omega_1,$$

where Ω is the domain of the problem (5.25).

Let $\{\Delta^{jk} = \Delta_1^{jk} \times \Delta_2^{jk}\}_{k=1}^\infty$ be a sequence of partitions of the rectangle $\overline{\Omega}_1 = [a_1, b_1] \times [a_2, b_2]$, $j = 1, 2$ and

$$
\begin{aligned}
&\Delta_1^{jk} : a_1 = x_{10}^{jk} < x_{11}^{jk} < \cdots < x_{1M_{jk}}^{jk} = b_1, \quad M_{jk} \to \infty \text{ at } k \to \infty,\\
&\Delta_2^{jk} : a_2 = x_{20}^{jk} < x_{21}^{jk} < \cdots < x_{2N_{jk}}^{jk} = b_2, \quad N_{jk} \to \infty \text{ at } k \to \infty.
\end{aligned} \tag{6.1}
$$

We set

$$
\begin{aligned}
&\Pi_1^{jk} = \max_{0 \le i \le M_{jk}-1} (x_{1(i+1)}^{jk} - x_{1i}^{jk}), \quad \underline{\Pi}_1^{jk} = \min_{0 \le i \le M_{jk}-1} (x_{1(i+1)}^{jk} - x_{1i}^{jk}),\\
&\Pi_2^{jk} = \max_{0 \le i \le N_{jk}-1} (x_{2(i+1)}^{jk} - x_{2i}^{jk}), \quad \underline{\Pi}_2^{jk} = \min_{0 \le i \le N_{jk}-1} (x_{2(i+1)}^{jk} - x_{2i}^{jk}),\\
&\Pi^{jk} = \max(\Pi_1^{jk}, \Pi_2^{jk}), \quad \underline{\Pi}^{jk} = \min(\underline{\Pi}_1^{jk}, \underline{\Pi}_2^{jk}).
\end{aligned} \tag{6.2}
$$

We suppose that there exists a positive constant σ such that

$$\frac{\Pi^{jk}}{\underline{\Pi}^{jk}} \le \sigma, \quad k \in \mathbb{N}, \quad j = 1, 2. \tag{6.3}$$

Define spaces W^{jk} as follows:

$$W^{jk} = S_{3i}(\Delta_1^{jk}, [a_1, b_1]) \bigotimes S_{3i}(\Delta_2^{jk}, [a_2, b_2]), \quad j = 1, 2, \quad i = 1 \text{ or } 2, \tag{6.4}$$

the symbol $\bigotimes$ denoting the tensor product.

The space W^{jk} consists of all functions h such that

$$h(x_1, x_2) = \sum_l u_l(x_1) v_l(x_2), \quad u_l \in S_{3i}(\Delta_1^{jk}, [a_1, b_1]), \quad v_l \in S_{3i}(\Delta_2^{jk}, [a_2, b_2]).$$

We mention that $W^{jk} \subset W_p^2(\Omega_1)$, $p > 1$, and the following condition is satisfied:

$$\lim_{k\to\infty} \inf_{h\in W^{jk}} ||v - h||_{W_p^2(\Omega_1)} = 0, \quad v \in W_p^2(\Omega_1), \quad j = 1, 2, \tag{6.5}$$

see [23], II, 9, III, 7.

Define spaces U_k as follows:

$$U_k \text{ is the space of restrictions of the elements of } \quad W^k = W^{1k} \times W^{2k} \text{ to } \Omega. \tag{6.6}$$

Let $\{\gamma_{ljk}\}_{l=1}^{J_{jk}}$ be the basis of W^{jk} that is formed by the multiplication of the one dimensional cardinal splines of the spaces $S_{3i}(\Delta_1^{jk}, [a_1, b_1])$ and $S_{3i}(\Delta_2^{jk}, [a_2, b_2])$. Since the supports of some functions γ_{ljk} and $\overline{\Omega}$ are mutually disjoint, the dimension of the space U_k can be significantly less than the dimension of W^k, especially at large k.

Taking (6.5) and [23], III, 3 into account, we obtain.

Lemma 6.1. *Let Δ_i^{jk} be defined by* (6.1) *and the condition* (6.3) *be satisfied. Then*

$$\lim_{k\to\infty} \inf_{h\in U_k} ||v - h||_{W_p^2(\Omega)^2} = 0, \quad v \in W_p^2(\Omega)^2, \quad p > 1. \tag{6.7}$$

If $i = 2$ in (6.4) *then*

$$\lim_{k\to\infty} \inf_{h\in U_k} ||v - h||_{W_p^{2+\alpha}(\Omega)^2} = 0, \quad v \in W_p^{2+\alpha}(\Omega)^2, \quad \alpha \in (0, 1). \tag{6.8}$$

6.2. **Approximation of the solution to the problem** (5.25)**.** The operator A, that is given by (5.22)–(5.24), is an isomorphism of $W_p^2(\Omega)^2$ onto

$$V = L_p(\Omega)^2 \times W_p^{2-\frac{1}{p}}(S_1)^2 \times W_p^{1-\frac{1}{p}}(S_2)^2.$$

The dual space V^* is defined by (5.38).

The spaces $V_k^* \subset V^*$, that satisfy the condition (2.2), could be formed by step-functions, given in Ω and on S_1, and S_2. Let $\{\psi_{rk}\}_{r=1}^{G_k}$ be a basis of V_k^*.

The Petrov-Galerkin approximation of the solution to the problem (5.25) is defined by

$$u_k \in U_k, \quad (Au_k, \psi_{rk}) = (f, \psi_{rk}), \quad r = 1, \ldots, G_k, \tag{6.9}$$

that is

$$P_k A u_k = P_k f. \tag{6.10}$$

Here $f = (K, \hat{u}, F)$ and P_k is the adjoint of the operator P_k^* that projects V^* onto V_k^*.

Since ψ_{rk} are step-functions, (Au_k, ψ_{rk}) takes the form of scalar product in $L_2(\Omega)^2 \times L_2(S_1)^2 \times L_2(S_2)^2$, and we have not to calculate fractional derivatives.

It follows from [4], Section 26.3 that the embeddings $W_p^{2+\alpha}(\Omega) \to W_p^2(\Omega)$ and $W_p^2(\Omega) \to W_p^{2-\alpha}(\Omega)$, $\alpha \in (0, 1)$, $p > 1$ are compact. Because of this, applying Theorem 2.1, Remarks 1 and 2 from Section 2, and Lemma 6.1, we obtain

Theorem 6.1. *Suppose that the conditions of Theorem 5.2 are satisfied. Let the spaces U_k be defined by* (6.6)*, and* (6.3) *be fulfilled. Let also $V_k^* = (A(U_k))^*$, $k \in \mathbb{N}$ and $(K, \hat{u}, F) \in V$. Then there exists a unique solution u_k to the problem*(6.9) *and $u_k \rightharpoonup u$ in $W_p^2(\Omega)^2$, $u_k \to u$ in $W_p^{2-\alpha}(\Omega)^2$, where u is the solution to the problem* (5.25) *and $\alpha \in (0,1)$. If, in addition, the conditions* (5.35) *and* (5.36) *are satisfied, S is of the class $C^{2,1}$, and $i = 2$ in* (6.4)*, then $u_k \to u$ in $W_p^2(\Omega)^2$.*

6.3. **The Petrov-Galerkin method for the operator $\mathbf{A^*}$.** We consider the following problem:

Given $g \in (W_p^2(\Omega)^2)^*$, find $v \in V^*$ such that

$$A^* v = g, \tag{6.11}$$

where A^* is the adjoint of the operator A, that is specified by (5.22)–(5.24), V^* is given by (5.38).

The Petrov-Galerkin approximation of the solution to the problem (6.11) is defined as follows:

$$v_k \in V_k^*, \quad (Ah, v_k) = (h, g), \quad h \in U_k, \tag{6.12}$$

where V_k^* and U_k are finite-dimensional subspaces of V^* and $U = W_p^2(\Omega)^2$.

The spaces U_k, which satisfy the condition (6.7), were defined above in 6.1. The spaces $V_k^* = (A(U_k))^*$ can be formed by step-functions. Then, by analogy with the above, we obtain that $v_k \rightharpoonup v$ in V^*.

6.4. **Approximation of a solution to the problem** (5.41)**.** Let U_k be defined by (6.6). Let $\{\varphi_{lk}\}_{l=1}^{G_k}$ be the basis of U_k, that is formed by the multiplication of the one-dimensional cardinal splines.

In line with (5.56) and (5.57), we set

$$\overset{\circ}{\varphi}_{lk} = \varphi_{lk} - \check{P}\varphi_{lk} = \varphi_{lk} - \sum_{i=1}^{3} (\varphi_{lk}, \tilde{w}_i)\tilde{w}_i, \quad l = 1, \dots, G_k. \tag{6.13}$$

Define $\overset{\circ}{U}_k$ as the span of the functions $\overset{\circ}{\varphi}_{lk}$. Since $N \subset U_k$, the functions $\overset{\circ}{\varphi}_{lk}$ are linearly dependent, and the dimension of the space $\overset{\circ}{U}_k$ equals $G_k - 3$.

At the conditions of Lemma 6.1, the spaces $\overset{\circ}{U}_k$ satisfy the condition (4.2). The spaces $\overset{\circ}{V}{}_k^* = (A(\overset{\circ}{U}_k))^* \subset \overset{\circ}{V}{}^*$ can be formed by step-functions. Let $\{\psi_{rk}\}_{r=1}^{G_k-3}$ be a basis of $\overset{\circ}{V}{}_k^*$.

The Petrov-Galerkin approximation of a solution to the problem (5.41) is defined as

$$u_k = \sum_{l=1}^{G_k-3} c_l \overset{\circ}{\varphi}_{lk}, \quad (Au_k, \psi_{rk}) = (f, \psi_{rk}), \quad r = 1, \dots, G_k - 3, \tag{6.14}$$

where $A = (A_1, A_2)$, $f = (K, F^1, F^2)$.

Applying the Theorem 4.2, we obtain.

Theorem 6.2. *Suppose that the conditions of Theorem 5.6 and Lemma 6.1 are satisfied. Let $\overset{\circ}{U}_k$ be the span of the functions $\overset{\circ}{\varphi}_{lk}$, which are defined* (6.13)*. Let $\{\psi_{rk}\}_{r=1}^{G_k-3}$ be a basis of $\overset{\circ}{V}{}_k^* = (A(\overset{\circ}{U}_k))^*$. Then for an arbitrary k there exists a unique solution u_k to the problem* (6.14) *and $u_k \rightharpoonup u$ in $W_p^2(\Omega)^2$, and $u_k \to u$ in $W_p^{2-\alpha}(\Omega)^2$, where $u \in \overset{\circ}{U}$ is a solution to the problem* (5.41) *and $\alpha \in (0,1)$. If, in addition, the conditions of Theorem 5.7 are satisfied and $i = 2$ in* (6.4)*, then $u_k \to u$ in $W_p^2(\Omega)^2$.*

By analogy with the set forth above, we can approximate solutions of three-dimensional problems of the elasticity theory. In this case, the space $W_p^2(\Omega)^3$ is approximated by the restriction to Ω of the spaces $W^{1k} \times W^{2k} \times W^{3k}$, where

$$W^{jk} = S_{3i}(\Delta_1^{jk}, [a_1, b_1]) \bigotimes S_{3i}(\Delta_2^{jk}, [a_2, b_2]) \bigotimes S_{3i}(\Delta_3^{jk}, [a_3, b_3]),$$
$$j = 1, 2, 3, \quad i = 1 \text{ or } 2, \quad \overline{\Omega} \subset [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3].$$

The spaces V_k^* could be constructed by step-functions.

7. **The Petrov-Galerkin method for a parabolic problem.**

7.1. **Parabolic problem.** We consider an operator A_1, that is given by

$$A_1 u = \frac{\partial u}{\partial t} + \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{2} a_i \frac{\partial u}{\partial x_i} + au \text{ in } Q = \Omega \times (0, T). \tag{7.1}$$

We suppose that $T < \infty$ and

$$a_{ij} \in C(\overline{Q}), \quad a_i, a \in L_\infty(Q), \tag{7.2}$$

$$\mu_1 |\gamma|^2 \le a_{ij}(x,t)\gamma_i\gamma_j \le \mu_2 |\gamma|^2, \quad (x,t) \in \overline{Q}, \quad \gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2, \tag{7.3}$$

μ_1 and μ_2 are positive constants.

We also assume that

(C7.1): Ω is a bounded domain in $\mathbb{R}^2$ with a boundary S of the class C^2.

Define a boundary operator and an operator of initial data as follows:

$$A_2 u = u \text{ on } S_T = S \times (0, T), \quad A_3 u = u \text{ on } \Omega \times \{t = 0\}. \tag{7.4}$$

We consider the problem: Find u satisfying

$$\begin{aligned} &A_1 u = f \text{ in } Q, \\ &A_2 u = \hat{u} \text{ on } S_T, \\ &A_3 u = u_0 \text{ on } \Omega. \end{aligned} \tag{7.5}$$

Set

$$\begin{aligned} &V_1 = \{g | g = (f, \hat{u}, u_0) \in L_p(Q) \times W_p^{2-\frac{1}{p}, 1-\frac{1}{2p}}(S_T) \times W_p^{2-\frac{2}{p}}(\Omega), \ p \ge 2\}, \\ &V = \{g | g = (f, \hat{u}, u_0) \in V_1, \quad u_0|_S = \hat{u}|_{t=0}\}. \end{aligned} \tag{7.6}$$

It follows from the embedding results, see [4], Chapter 5, Section 24, that if $w_n \to w$ in $W_p^{2-\frac{2}{p}}(\Omega)$, $h_n \to h$ in $W_p^{2-\frac{1}{p}, 1-\frac{1}{2p}}(S_T)$, $p \ge 2$, and $w_n|_S = h_n|_{t=0}$, then $w|_S = h|_{t=0}$. Therefore, the set V, provided with the norm

$$||(f, \hat{u}, u_0)||_V = ||f||_{L_p(Q)} + ||\hat{u}||_{W_p^{2-\frac{1}{p}, 1-\frac{1}{2p}}(S_T)} + ||u_0||_{W_p^{2-\frac{2}{p}}(\Omega)}, \tag{7.7}$$

is a Banach space.

The next result follows from [9], IV, 9.

Theorem 7.1. *Suppose that the conditions* $(C7.1)$, (7.2), *and* (7.3) *are satisfied. Then for any triple* $(f, \hat{u}, u_0) \in V$, *there exists a unique* $u \in W_p^{2,1}(Q)$ *that satisfies the conditions* (7.5) *and there exists* $c > 0$ *such that*

$$||u||_{W_p^{2,1}(Q)} \le c ||(f, \hat{u}, u_0)||_V, \quad (f, \hat{u}, u_0) \in V. \tag{7.8}$$

The operator $A = (A_1, A_2, A_3)$ *is an isomorphism of* $W_p^{2,1}(Q)$ *onto* V.

7.2. **Spaces $\mathbf{U_k}$, $\mathbf{V_k^*}$ and the Petrov-Galerkin approximations.** Let Ω_1 be a rectangular domain in $\mathbb{R}^2$ such that

$$\Omega_1 = \{x | x = (x_1, x_2),\ \ a_1 < x_1 < b_1,\ \ a_2 < x_2 < b_2\},\ \ \Omega \subset \Omega_1,$$

where Ω is the domain of the problem (7.5).

Let $\{\Delta^k = \Delta_1^k \times \Delta_2^k\}_{k=1}^{\infty}$ be a sequence of partitions of the rectangle $\overline{\Omega}_1$, where

$$\begin{aligned} &\Delta_1^k : a_1 = x_{10}^k < x_{11}^k < \cdots < x_{1M_k}^k = b_1, \quad M_k \to \infty \ \text{ at } k \to \infty, \\ &\Delta_2^k : a_2 = x_{20}^k < x_{21}^k < \cdots < x_{2N_k}^k = b_2, \quad N_k \to \infty \ \text{ at } k \to \infty. \end{aligned} \tag{7.9}$$

Let also $\{\Delta_t^k\}_{k=1}^{\infty}$ be a sequence of partitions of the segment $[0, T]$,

$$\Delta_t^k : 0 = t_0^k < t_1^k < \cdots < t_{L_k}^k = T, \quad L_k \to \infty \ \text{ at } k \to \infty. \tag{7.10}$$

We set

$$\begin{aligned} &\Pi_1^k = \max_{0 \le i \le M_k - 1} (x_{1(i+1)}^k - x_{1i}^k), \quad \underline{\Pi}_1^k = \min_{0 \le i \le M_k - 1} (x_{1(i+1)}^k - x_{1i}^k), \\ &\Pi_2^k = \max_{0 \le i \le N_k - 1} (x_{2(i+1)}^k - x_{2i}^k), \quad \underline{\Pi}_2^k = \min_{0 \le i \le N_k - 1} (x_{2(i+1)}^k - x_{2i}^k), \\ &\Pi_3^k = \max_{0 \le i \le L_k - 1} (t_{i+1}^k - t_i^k), \quad \underline{\Pi}_3^k = \min_{0 \le i \le L_k - 1} (t_{i+1}^k - t_i^k), \\ &\Pi^k = \max(\Pi_1^k, \Pi_2^k, \Pi_3^k), \quad \underline{\Pi}^k = \min(\underline{\Pi}_1^k, \underline{\Pi}_2^k, \underline{\Pi}_3^k). \end{aligned} \tag{7.11}$$

We assume that there exists a positive constant σ_1 such that

$$\frac{\Pi^k}{\underline{\Pi}^k} \le \sigma_1, \quad k \in \mathbb{N}. \tag{7.12}$$

Define a space X^k in the form

$$X^k = S_{3i}(\Delta_1^k, [a_1, b_1]) \bigotimes S_{3i}(\Delta_2^k, [a_2, b_2]), \quad i = 1 \ \text{ or } 2. \tag{7.13}$$

Denote

$$Y^k \text{ is the set of restrictions of the elements of } \ X^k \text{ to } \Omega. \tag{7.14}$$

Let $S_1(\Delta_t^k, [0, T])$ be the space of splines of the first degree, that consists of functions w such that $w \in C([0.T])$, and on each subsegment $[t_i^k, t_{i+1}^k]$, the function w is affine, i.e. it is a polynomial of the first degree.

We define subspaces U_k of $W_p^{2,1}(Q)$ as follows:

$$U_k = Y^k \bigotimes S_1(\Delta_t^k, [0, T]). \tag{7.15}$$

Let $\{e_{lk}\}_{l=1}^{E_k}$ be the basis of Y_k that is formed by the multiplication of the one-dimensional cardinal splines. Let $\{z_{ik}\}_{i=1}^{J_k}$ be the basis of $S_1(\Delta_t^k, [0, T])$, which is formed by the cardinal splines. The dimension of the space $S_1(\Delta_t^k, [0, T])$ is equal to the number of points of partition of $[0, T]$, i.e. $J_k = L_k + 1$, see (7.10).

The set of functions

$$e_{lk}(x) z_{ik}(t), \quad l = 1, \ldots, E_k, \quad i = 1, \ldots, J_k \tag{7.16}$$

is a basis of U_k.

Define functions φ_{jk}, $j = 1, \dots, J_k E_k$ as follows:

$$\varphi_{jk}(x,t) = \begin{cases} e_{jk}(x) z_{1k}(t) \quad \text{at } 1 \le j \le E_k, \\ e_{j-E_k}(x) z_{2k}(t) \quad \text{at } E_k + 1 \le j \le 2E_k, \\ \dots\dots\dots \qquad \dots\dots\dots \qquad \dots\dots\dots \\ e_{j-(J_k-1)E_k}(x) z_{J_k k}(t) \quad \text{at } (J_k - 1)E_k + 1 \le j \le J_k E_k. \end{cases} \tag{7.17}$$

Theorem 7.2. *Suppose that the condition* (7.12) *is satisfied and the spaces* U_k *are defined by* (7.15)*. Then the functions* φ_{jk} *defined by* (7.17) *form a basis of* U_k*, and*

$$\lim_{k\to\infty} \inf_{h\in U_k} ||v - h||_{W_p^{2,1}(Q)} = 0, \quad v \in W_p^{2,1}(Q), \quad p \ge 2. \tag{7.18}$$

Proof. Let $Q_1 = (a_1, b_1) \times (a_2, b_2) \times (0, T)$. Denote the set of continuous in $\overline{Q}_1$ functions, which have continuous derivatives in $\overline{Q}_1$ up to the third order with respect to x and the second order with respect to t, by $C^{3,2}(\overline{Q}_1)$. The space $C^{3,2}(\overline{Q}_1)$ is tightly embedded in $W_p^{2,1}(Q_1)$, $p \ge 2$.

Let $y \in C^{3,2}(\overline{Q}_1)$ and the function $S(y) \in X^k \bigotimes S_1(\Delta_t^k, [0,T])$ interpolates y on the grid $\Delta_1^k \times \Delta_2^k \times \Delta_t^k$, according to the orders and the defects of the one-dimensional splines, see [23], II, 1, II, 2, II, 9, III, 7. Then we have

$$||y - S(y)||_{W_p^{2,1}(Q_1)} \le c\Pi^k, \quad k \in \mathbb{N}, \tag{7.19}$$

where c depends on y but independent of k.

Since $C^{3,2}(\overline{Q}_1)$ is tightly embedded in $W_p^{2,1}(Q_1)$, we obtain (7.18) from (7.19). □

The dual space of V_1 is defined by

$$V_1^* = \{w | w = (w_1, w_2, w_3),\ w_1 \in L_q(Q),\ w_2 = \left(W_p^{2-\frac{1}{p}, 1-\frac{1}{2p}}(S_T)\right)^*,\ w_3 \in \left(W_p^{2-\frac{2}{p}}(\Omega)\right)^*, \\ q = \frac{p}{p-1},\ p \ge 2\}. \tag{7.20}$$

The spaces $\left(W_p^{2-\frac{1}{p}, 1-\frac{1}{2p}}(S_T)\right)^*$ and $\left(W_p^{2-\frac{2}{p}}(\Omega)\right)^*$ can be determined as the completion of $L_2(S_T)$ and $L_2(\Omega)$ in the norms

$$||v||_1 = \sup |(v,h)_{L_2(S_T)}|, \quad ||h||_{W_p^{2-\frac{1}{p}, 1-\frac{1}{2p}}(S_T)} = 1,$$
$$||z||_2 = \sup |(z,e)_{L_2(\Omega)}|, \quad ||e||_{W_p^{2-\frac{2}{p}}(\Omega)} = 1.$$

Therefore, the set of the products of step-functions given in Q and on S_T and Ω is dense in V_1^*, and finite-dimensional subspaces $V_k^* = (A(U_k))^*$ can be formed by step-functions. Let $\{\psi_{rk}\}_{r=1}^{J_k E_k}$ be a basis of V_k^*.

Approximate solution to the problem (7.5) is defined by

$$u_k = \sum_{j=1}^{J_k E_k} c_j \varphi_{jk}, \quad (Au_k, \psi_{rk}) = (g, \psi_{rk}), \quad r = 1, \dots, J_k E_k, \tag{7.21}$$

where $g = (f, \hat{u}, u_0)$, $A = (A_1, A_2, A_3)$.

Applying the Theorems 2.1, 7.1, and 7.2, we obtain.

Theorem 7.3. *Suppose that the conditions of the Theorem 7.1 are satisfied and* $(f, \hat{u}, u_0) \in V$*. Let the spaces* U_k *be defined by* (7.15) *and* (7.12) *holds. Let also* $V_k^* = (A(U_k))^*$ *for all* k*. Then for an arbitrary* k *there exists a unique solution* u_k *to the problem* (7.21) *and* $u_k \rightharpoonup u$

in $W_p^{2,1}(Q)$ *and* $u_k \to u$ *in* $W_p^{2-\alpha,1-\frac{1}{2}\alpha}(Q)$, $\alpha \in (0,1)$, *where* u *is the solution to the problem* (7.5).

8. **Nonlinear problems.**

8.1. **Operators and problems under consideration.** We suppose that A is a nonlinear mapping of U to V, U and V being separable reflexive Banach spaces, and the following conditions are satisfied.

(C8.1): A is a Fréchet continuously differentiable mapping of U to V. At an arbitrary point $v \in U$, the Fréchet derivative $A'(v)$ is invertible, i.e. there exists the inverse operator $(A'(v))^{-1}$ of $A'(v)$, and $(A'(v))^{-1} \in \mathcal{L}(V,U)$. There is $u_0 \in U$ such that $A(u_0) = 0$.

(C8.2): A is a restriction of an operator $\tilde{A}$ which is a one-to-one mapping of W to W^1, where W and W^1 are separable reflexive Banach spaces such that $U \subset W$, $V \subset W^1$; in this case,

$$A(u) = \tilde{A}(u), \quad u \in U.$$

We consider the problem. Given $f \in V$, find u satisfying

$$u \in U, \quad A(u) = f. \tag{8.1}$$

Theorem 8.1. *Suppose that the condition* $(C8.1)$ *is satisfied. Then for an arbitrary* $f \in V$, *there exists a solution to the problem* (8.1). *If, in addition, the condition* $(C8.2)$ *holds, then the operator* A *is a homomorphism of* U *onto* V.

Proof. For a given $f \in V$, we define a mapping $M : [0,1] \times U \to V$ as follows:

$$M(e,v) = A(v) - ef. \tag{8.2}$$

We consider the problem: Given $e \in [0,1]$, find $u_e \in U$ such that

$$M(e,u_e) = 0. \tag{8.3}$$

Then the problem (8.1) is represented in the form

$$M(1,u_1) = 0, \tag{8.4}$$

i.e. the element $u = u_1$ is a solution to the problem (8.1).

At an arbitrary fixed $e \in [0,1]$, the partial derivative

$$\frac{\partial M}{\partial v}(e,v) = A'(v) \tag{8.5}$$

is an isomorphism of U onto V, and $M(0,u_0) = A(u_0) = 0$. This enables us to use the implicit function theorem, see e.g. [18], Chapter 3, Section 8. Then we obtain, that there is $\varepsilon_1 > 0$ such that for any $e \in [0,e_1]$ there exists a unique $u_e \in U$ that satisfies (8.3). Moreover, the function $g : e \to g(e) = u_e$ is a continuously differentiable mapping of $[0,e_1]$ into U.

The mapping $\frac{\partial M}{\partial v}(e_1,u_{e_1}) = A'(u_{e_1})$ is an isomorphism of U onto V. Therefore, the solution to the problem (8.3) can be prolonged to a point $e_2 > e_1$. In the same manner, the solution of the problem (8.3) can be prolonged up to $e = 1$.

Therefore, there exists a solution to the problem (8.1) for any $f \in V$. If the condition $(C8.2)$ holds, then any solution to the problem (8.1) also is a solution to the problem $\tilde{A}(u) = f$, and hence, the solution to the problem (8.1) is unique.

Thus, A is a bijection of U onto V.

Define a mapping $g_1 : (U \times V) \to V$ as follows:

$$g_1(u, f) = A(u) - f. \tag{8.6}$$

For any f there exists a unique $u(f)$ such that $g_1(u(f), f) = 0$. Since the operator

$$\frac{\partial g_1}{\partial u}(u(f), f) = A'(u(f)) \tag{8.7}$$

is invertible, we obtain from the implicit function theorem that $f \to u(f)$ is a continuous mapping of V to U. Therefore, A is a homomorphism of U onto V. $\square$

8.2. **Approximation of the solution to the problem** (8.1)**.** We apply the modified Newton method for computing the Petrov-Galerkin approximations of the solution to the problem (8.1).

Let U^1 and V^1 be a pair of separable reflexive Banach spaces which satisfy the condition

$$\begin{gathered} U \subset U^1 \subset W, \quad V \subset V^1 \subset W^1, \text{ each space is dense} \\ \text{in the consequent, the embedding } \; V \to V^1 \; \text{ is compact.} \end{gathered} \tag{8.8}$$

We suppose

(C8.3): A is a twice Fréchet continuously differentiable mapping of U to V and U^1 to V^1, and at arbitrary points of U and U^1, the Fréchet derivatives are invertible, i.e.

$$(A'(u))^{-1} \in \mathcal{L}(V, U), \quad u \in U, \quad (A'(w))^{-1} \in \mathcal{L}(V^1, U^1), \quad w \in U^1.$$

The functions $u \to ||A'(u)||_{\mathcal{L}(U,V)}$ and $u \to ||A''(u)||_{\mathcal{L}_2(U,U;V)}$ are bounded in any ball of U, and $A(0) = 0$.

Let $\{U_k\}$ be a sequence of finite-dimensional subspaces of U, which satisfy (2.1). Let $\{V^*_{k0}\}_{k=1}^{\infty}$ be a sequence of subspaces of V^* such that $V^*_{k0} = (A'(0)(U_k))^*$. Let P^*_{k0} be a projection of V^* onto V^*_{k0} and $P_{k0} = P^{**}_{k0}$. Then we have

$$||(P_{k0}A'(0))^{-1}||_{\mathcal{L}(V_{k0},U_k)} \leq \gamma, \quad \gamma > 0, \quad k \in \mathbb{N}, \tag{8.9}$$

where $V_{k0} = A'(0)(U_k)$, and γ depends on V^*_{k0}.

We also have

$$P_{k0}A(0) = 0. \tag{8.10}$$

Taking (C8.3), (8.9), and (8.10) into account, we obtain from [7], XVIII, 1.5 that there exists $\varepsilon_1 > 0$ such that for any n there is a unique solution to the problem

$$u^0_{k(n+1)} \in U_k, \quad P_{k0}A'(0)(u^0_{k(n+1)} - u^0_{kn}) = -P_{k0}(A(u^0_{kn}) - \varepsilon_1 f), \quad u^0_{k0} = 0, \tag{8.11}$$

and $u^0_{kn} \to u^0_k$, where u^0_k is the solution to the problem

$$P_{k0}\, A(u^0_k) = \varepsilon_1 P_{k0}\, f, \quad k \in \mathbb{N}. \tag{8.12}$$

Next, we have to prolong the Petrov-Galerkin approximation (8.12) by using the modified Newton method. By (8.9) we can consider that at small ε_1, there exists $\gamma_0 > 0$ such that

$$||(P_{k0}\, A'(u^0_k))^{-1}||_{\mathcal{L}(A'(u^0_k)(U_k),U_k)} \leq \gamma_0. \tag{8.13}$$

However, we cannot estimate γ_0, it may be smaller as well as larger than γ. If γ_0 is very large, it is desirable to find $V^*_{k1} \subset V^*$, and $u^1_{k0} \subset U_k$, such that

$$(A(u^1_{k0}) - \varepsilon_1 f,\, P^*_{k1}\, h) = 0, \quad h \in V^*, \tag{8.14}$$

where P^*_{k1} is a projection of V^* onto V^*_{k1}, and

$$||(P_{k1}\, A'(u^1_{k0}))^{-1}||_{\mathcal{L}(V_{k1},U_k)} \leq \gamma_1, \quad \gamma_1 < \gamma_0, \quad k \in \mathbb{N}, \tag{8.15}$$

where $P_{k1} = P^{**}_{k1}$ and $V_{k1} = A'(u^1_{k0})(U_k)$.

Thus, we face the following problem:

Problem B: Given the subspace $V^*_{k0} \subset V^*$ and $u^0_k \in U_k$, which satisfy (8.12) and (8.13), where $P_{k0} = P^{**}_{k0}$ and P^*_{k0} is a projection of V^* onto V^*_{k0}. Find $V^*_{k1} \subset V^*$ and $u^1_{k0} \in U_k$ such that (8.14) and (8.15) hold.

There exists a solution to the problem B that can be computed, see Subsection 8.3 below.

It follows from [7], XVIII, 1.5 that there exists $\varepsilon_2 > 0$ such that there is a unique solution to the problem

$$\begin{gathered} u^1_{k(n+1)} \in U_k, \quad P_{k1}A'(u^1_{k0})(u^1_{k(n+1)} - u^1_{kn}) = -P_{k1}(A(u^1_{kn}) - (\varepsilon_1+\varepsilon_2)f), \\ n = 0,1,2,\ldots, \quad \varepsilon_1+\varepsilon_2 \leq 1, \end{gathered} \tag{8.16}$$

and $u^1_{kn} \to u^1_k$, where u^1_k is the solution to the problem

$$u^1_k \in U_k, \quad P_{k1}A(u^1_k) = (\varepsilon_1+\varepsilon_2)P_{k1}f. \quad k \in \mathbb{N}. \tag{8.17}$$

We mention, that if γ_0 is not very large, then we take $P_{k1} = P_{k0}$ in (8.16), and we obtain (8.17) with $P_{k1} = P_{k0}$.

By analogy, at some finite l, we obtain

$$u_k \in U_k, \quad P_{kl}A(u_k) = P_{kl}f, \quad k \in \mathbb{N}. \tag{8.18}$$

Theorem 8.2. *Suppose that the operator A satisfies the conditions ($C8.2$) and ($C8.3$), and also (8.8) holds. Let $f \in V$ and $\{U_k\}$ be a sequence of finite-dimensional subspaces of U which comply with (2.1). Let also $\{u_k\}$ be a sequence of solutions to the problem (8.18). Then $u_k \to u$ in U^1, where u is the solution to the problem (8.1)*

Proof. Since $A'(w)$ is an isomorphism of U onto V for any $w \in U$, and (2.1) holds, we obtain from (8.18) that

$$\lim(A(u_k), h) = (f, h), \quad h \in V^*, \tag{8.19}$$

that is

$$A(u_k) \rightharpoonup f \ \text{ in } V. \tag{8.20}$$

(8.8) yields

$$A(u_k) \to f \ \text{ in } V^1. \tag{8.21}$$

It follows from Theorem 8.1 that the operator A is a homomorphism of U onto V and U^1 onto V^1. Because of this, (8.21) implies

$$u_k \to u \ \text{ in } U^1, \quad \lim A(u_k) = A(u) \ \text{ in } V^1. \tag{8.22}$$

and

$$A(u) = f. \tag{8.23}$$

As $f \in V$, we obtain $u \in U$. $\square$

8.3. **To the problem B.** We consider a more extended problem than the problem B. Let $\{\varphi_{ik}\}_{i=1}^{N_k}$ be a basis of U_k, and $\{\psi_{ik}^0\}_{i=1}^{N_k}$ be a basis of V_{k0}^*. We can reckon that

$$||\psi_{ik}^0||_{V^*} = 1, \quad (A'(0)\varphi_{jk}, \psi_{ik}^0) = \zeta_{ik}^0 \delta_{ji}, \quad \zeta_{ik}^0 > 0, \quad i, j, = 1, \dots, N_k. \tag{8.24}$$

Define the following set:

$$\begin{aligned} G_\beta = \{\psi | \psi = \{\psi_{ik}\}_{i=1}^{N_k},\ ||\psi_{ik}||_{V^*} = 1,\ (A'(0)\varphi_{jk}, \psi_{ik}) = \zeta_{ik}, \delta_{ij}, \\ \alpha \le \zeta_{ik} \le \tilde{\zeta}_{ik},\ |\zeta_{ik} - \zeta_{ik}^0| \le \beta\}, \end{aligned} \tag{8.25}$$

where α and β are small positive constants, and

$$\tilde{\zeta}_{ik} = \sup_{h \in W_{ik}} (A'(0)\varphi_{ik}, h), \tag{8.26}$$

where

$$W_{ik} = \{h | h \in V^*,\ ||h||_{V^*} = 1,\ (A'(0)\varphi_{jk}, h) = 0,\ j = 1, \dots, i-1, i+1, \dots, N_k\}. \tag{8.27}$$

The metric in G_β is defined as follows: If $\psi^1 = \{\psi_{ik}^1\}_{i=1}^{N_k}$ and $\psi^2 = \{\psi_{ik}^2\}_{i=1}^{N_k}$ are elements of G_β, then the distance is given by

$$\rho(\psi^1, \psi^2) = \max |\zeta_{ik}^1 - \zeta_{ik}^2|, \quad i = 1, \dots, N_k, \tag{8.28}$$

where

$$\zeta_{ik}^l = (A'(0)\varphi_{ik}, \psi_{ik}^l), \quad l = 1 \text{ or } 2. \tag{8.29}$$

For a given $\psi \in G_\beta$, we define a mapping $g_\psi : [0,1] \times U_k \to \mathbb{R}^{N_k}$ as follows:

$$g_\psi(\lambda, u) = \begin{cases} (A(u) - \varepsilon_1 f,\ \lambda\psi_{1k} + (1-\lambda)\psi_{1k}^0) \\ \dots\dots\dots\dots \qquad \dots\dots \\ (A(u) - \varepsilon_1 f,\ \lambda\psi_{N_k k} + (1-\lambda)\psi_{N_k k}^0). \end{cases} \tag{8.30}$$

We consider the problem: Find a pair λ, u_λ satisfying

$$g_\psi(\lambda, u_\lambda) = 0, \quad \lambda \in [0,1]. \tag{8.31}$$

In the same way, that was used in the proof of Theorem 8.1, we obtain by the implicit function theorem, that at small β, there exists a unique solution to the problem (8.31) for any $\lambda \in [0,1]$, and the operator $\frac{\partial g_\psi}{\partial u}(1, u_1)$ is invertible.

The function u_1 depends on ψ. Because of this, we denote

$$u(\psi) = u_1. \tag{8.32}$$

It is apparent that $u(\psi)$ is the solution to the problem

$$u(\psi) \in U_k, \quad (A(u(\psi)) - \varepsilon_1 f, \psi_{ik}) = 0, \quad i = 1, \dots, N_k. \tag{8.33}$$

Let $V_{k\psi}^*$ be the span of the elements $\psi_{1k}, \dots, \psi_{N_k k}$. Let $P_{k\psi}^*$ be a projection of V^* onto $V_{k\psi}^*$ and $P_{k\psi}$ be the adjoint of $P_{k\psi}^*$.

It follows from (8.33) that

$$P_{k\psi}(A(u(\psi)) - \varepsilon_1 f) = 0. \tag{8.34}$$

We define a functional Ψ of the form

$$\Psi(\psi) = ||(P_{k\psi} A'(u(\psi)))^{-1}||_{\mathcal{L}(V_{k\psi}, U_k)}, \quad \psi \in G_\beta, \tag{8.35}$$

where $V_{k\psi} = A'(u(\psi))(U_k)$.

The functional Ψ is well-defined at a small β.

We consider the problem: Find $\tilde{\psi}$ satisfying

$$\tilde{\psi} \in G_\beta, \quad \Psi(\tilde{\psi}) = \min_{\psi \in G_\beta} \Psi(\psi). \tag{8.36}$$

The set G_β, being supplied with the metric (8.28), is a compact, and Ψ is continuous mapping of G_β into $\mathbb{R}$. Therefore, there exists a solution to the problem (8.36).

The element $\tilde{\psi}$ can be considered as a solution to the Problem B. We can also obtain a better solution to the problem B by using the above computations in which $\psi^0 = \{\psi^0_{ik}\}_{i=1}^{N_k}$ is changed for $\tilde{\psi} = \{\tilde{\psi}_{ik}\}_{i=1}^{N_k}$ and $A'(0)$ is changed for $A'(\tilde{\psi})$ in the above formulas.

9. **A problem on deformation of an elastoplastic body.**

9.1. **Main equations. Operators** A **and** $\tilde{A}$**.** At small deformations, the components of the stress tensor in an elastoplastic body are defined by the following formula, see [12], 5.12.

$$\sigma_{ij}(u) = \beta\varepsilon(u)\delta_{ij} + 2G(I(u))e_{ij}(u), \quad i, j = 1, 2, 3. \tag{9.1}$$

Here $u = (u_1, u_2, u_3)$ is the vector of displacements, β the compression modulus,

$$\varepsilon(u) = \frac{1}{3} \operatorname{div} u, \tag{9.2}$$

$e_{ij}(u)$ are the components of the deviator of the strain tensor

$$e_{ij}(u) = \varepsilon_{ij}(u) - \varepsilon(u)\delta_{ij}, \tag{9.3}$$

$\varepsilon_{ij}(u)$ are the components of the strain tensor that are defined by (5.2), $I(u)$ is the second invariant of the deviator of the strain tensor

$$I(u) = \sum_{i,j=1}^{3} (e_{ij}(u))^2, \tag{9.4}$$

G is plasticity modulus, which depends on $I(u)$.

The equations of equilibrium are defined by (5.5), where $i, j = 1, 2, 3$. Considering (9.1) and (9.2), these equations take the form

$$A_{1i}(u) = -\frac{1}{3}\,\beta\frac{\partial}{\partial x_i} \operatorname{div} u - 2\frac{\partial}{\partial x_j}(G(I(u))e_{ij}(u)) = K_i \text{ in } \Omega, \quad i = 1, 2, 3, \tag{9.5}$$

where K_i are components of the volume force vector $K = (K_1, K_2, K_3)$.

We prescribe displacements q on the boundary S of Ω,

$$A_2(u) = u|_S = q. \tag{9.6}$$

We suppose

(C9.1): The function G is a four times continuously differentiable mapping of $\mathbb{R}_+$ into $\mathbb{R}_+$ and there exist positive numbers $a_1 - a_4$ such that, at any $y \in \mathbb{R}_+$ the following inequalities hold:

$$a_2 \geq G(y) \geq a_1, \tag{9.7}$$

$$G(y) + 2\,\frac{dG}{dy}(y)y \geq a_3, \tag{9.8}$$

$$\left|\frac{dG}{dy}(y)y\right| \leq a_4. \tag{9.9}$$

The inequality (9.7) indicates that the plasticity modulus is bounded above and below by positive constants. The estimate (9.8) implies that under a simple shear, the stress increases as the strain increases. The inequality (9.9) is a restriction on $\frac{dG}{dy}(y)$ for large values of y. The inequalities (9.7)–(9.9) are natural from the physical viewpoint.

We also suppose

(C9.2): Ω is a bounded domain in $\mathbb{R}^3$ with a boundary S of the class C^3.

$$\beta \text{ is a positive constant.} \tag{9.10}$$

We denote

$$A_1 = (A_{11}, A_{12}, A_{13}), \quad A = (A_1, A_2), \tag{9.11}$$

where A_{1i} and A_2 are the operators defined in (9.5), (9.6).

We consider that the operator A has the following domain of definition

$$\mathcal{D}(A) = W_p^{2+\alpha}(\Omega)^3, \quad \alpha \in (0,1), \quad \alpha p > 3. \tag{9.12}$$

The norm in $W_p^{l+\alpha}(\Omega)$, l being a positive integer, is given by

$$||h||_{W_p^{l+\alpha}(\Omega)} = \Big(||h||^p_{W_p^l(\Omega)} + \sum_{|k|=l} \int_\Omega \int_\Omega \frac{|\mathcal{D}^k h(x) - \mathcal{D}^k h(y)|^p}{|x-y|^{3+p\alpha}}\, dx\, dy \Big)^{\frac{1}{p}}. \tag{9.13}$$

Let $\tilde{A} = (\tilde{A}_1, \tilde{A}_2)$ be an extension of the operator $A = (A_1, A_2)$ such that

$$\mathcal{D}(\tilde{A}) = H^1(\Omega)^3, \quad \tilde{A}(v) = A(v), \quad v \in W_p^{2+\alpha}(\Omega)^3. \tag{9.14}$$

9.2. **Problem for the operator $\tilde{A}$.** .

We assume that

$$K = (K_1, K_2, K_3) \in H^{-1}(\Omega)^3, \tag{9.15}$$

$$q = (q_1, q_2, q_3) \in H^{\frac{1}{2}}(S)^3. \tag{9.16}$$

Theorem 9.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a Lipschitz continuous boundary S. Suppose that the condition* (C9.1) *and* (9.10), (9.15), *and* (9.16) *are satisfied. Then there exists a unique u such that*

$$u \in H^1(\Omega)^3, \quad \tilde{A}_1(u) = K, \quad \tilde{A}_2(u) = q. \tag{9.17}$$

Proof. It follows from (9.16) that there exists a function w satisfying

$$w \in H^1(\Omega)^3, \quad w|_S = q. \tag{9.18}$$

We define a mapping L: $H_0^1(\Omega)^3 \to H^{-1}(\Omega)^3$ in the form

$$(L(v), h) = 3\beta \int_\Omega \varepsilon(v)\, \varepsilon(h)\, dx + 2\int_\Omega G(I(v+w))\, e_{ij}(v+w)\, e_{ij}(h)\, dx,$$
$$v, h \in H_0^1(\Omega)^3. \tag{9.19}$$

Consider the problem: Find g such that

$$g \in H_0^1(\Omega)^3, \quad (L(g), h) = (K, h) - 3\beta \int_\Omega \varepsilon(w)\, \varepsilon(h)\, dx, \quad h \in H_0^1(\Omega)^3. \tag{9.20}$$

Taking account that,

$$e_{ij}(h)\delta_{ij} = \sum_{i=1}^3 \Big(\frac{\partial h_i}{\partial x_i} - \frac{1}{3}\,\mathrm{div}\, h \delta_{ii} \Big) = 0, \tag{9.21}$$

we obtain by the Green formula, that if u is a solution to the problem (9.17), then $g = u - w$ is a solution to the problem (9.20). On the contrary, if g is a solution to the problem (9.20), then $u = g + w$ is a solution to the problem (9.17).

Under the conditions (C9.1) and (9.18), the operator L is a strongly monotone, coercive, and continuous mapping of $H_0^1(\Omega)^3$ to $H^{-1}(\Omega)^3$, see [13]. Therefore, there exists a unique solution to the problem (9.20). If w^1 is another function that satisfies (9.18), and g^1 is a solution of the problem (9.20) in which w is changed for w^1, then $g + w = g^1 + w^1$.

Therefore the solution of the problem (9.17) is unique. □

9.3. **Two lemmas.**

Lemma 9.1. *Suppose that the conditions (C9.1), (C9.2), and (9.10) are satisfied. Then the operator A is a twice Fréchet continuously differentiable mapping of $W_p^{2+\alpha}(\Omega)^3$ into $W_p^{\alpha}(\Omega)^3 \times W_p^{2+\alpha-\frac{1}{p}}(S)^3$, where $\alpha \in (0,1)$, $\alpha p > 3$. The derivatives of A are defined by*

$$
\begin{aligned}
& A_1'(v) = (A_{11}'(v), A_{12}'(v), A_{13}'(v)), && (9.22)\\
& A_2'(v)h = h|_S, \quad v, h \in W_p^{2+\alpha}(\Omega)^3, && (9.23)\\
& A_{1i}'(v)h = -\frac{1}{3}\beta\frac{\partial}{\partial x_i}\operatorname{div} h - 2\frac{\partial}{\partial x_j}[G(I(v))e_{ij}(h)] \\
& - 4\frac{\partial}{\partial x_j}\Big[\frac{dG}{dy}(I(v))e_{km}(v)\, e_{ij}(v)\, e_{km}(h)\Big] \text{ in } \Omega, \quad i,j,k,m = 1,2,3, && (9.24)\\
& A_2''(v) = 0, && (9.25)\\
& A_{1i}''(v)(h,z) = -4\frac{\partial}{\partial x_j}\Big[\frac{dG}{dy}(I(v))e_{km}(v)\, e_{ij}(h)e_{km}(z)\Big] \\
& - 8\frac{\partial}{\partial x_j}\Big[\frac{d^2G}{dy^2}(I(v))e_{lr}(v)\, e_{km}(v)\, e_{ij}(v)\, e_{km}(h)\, e_{lr}(z)\Big] \\
& - 4\frac{\partial}{\partial x_j}\Big[\frac{dG}{dy}(I(v))(e_{ij}(v)\, e_{km}(h)\, e_{km}(z) + e_{km}(v)\, e_{km}(h)\, e_{ij}(z))\Big], \\
& v, h, z \in W_p^{2+\alpha}(\Omega)^3, \quad i,j,k,m,l,r = 1,2,3. && (9.26)
\end{aligned}
$$

The functions

$$
\begin{aligned}
& v \to ||A_1'(v)||_{\mathcal{L}(W_p^{2+\alpha}(\Omega)^3, W_p^{\alpha}(\Omega)^3)}, \\
& v \to ||A_1''(v)||_{\mathcal{L}_2(W_p^{2+\alpha}(\Omega)^3, W_p^{2+\alpha}(\Omega)^3; W_p^{\alpha}(\Omega)^3)}
\end{aligned}
$$

are bounded in any ball of $W_p^{2+\alpha}(\Omega)^3$, and $A(0) = 0$.

Proof. It follows from the embedding theorem that

$$
W_p^{\alpha}(\Omega) \subset C^{\lambda}(\overline{\Omega}), \quad 0 < \lambda < \alpha - \frac{3}{p} \text{ at } \alpha p > 3. \tag{9.27}
$$

Taking (9.13) and (9.27) into account, we obtain

$$
\text{the function } f, g \to fg \text{ is a bilinear continuous mapping of } W_p^{1+\alpha}(\Omega) \times W_p^{1+\alpha}(\Omega) \text{ into } W_p^{1+\alpha}(\Omega), \tag{9.28}
$$

$$
\text{the function } f, g \to fg \text{ is a bilinear continuous mapping of } W_p^{1+\alpha}(\Omega) \times W_p^{\alpha}(\Omega) \text{ into } W_p^{\alpha}(\Omega). \tag{9.29}
$$

It is not difficult to verify that

$$\lim_{\gamma\to 0}\frac{(A_{1i}(v+\gamma h))(x)-(A_{1i}(v))(x)}{\gamma}=(A'_{1i}(v)h)(x),\quad x\in\Omega,$$
$$v,h\in W_p^{2+\alpha}(\Omega)^3,\quad i=1,2,3,\tag{9.30}$$

where A'_{1i} is given by (9.24).

(9.27)–(9.29) imply that $A'_{1i}\in\mathcal{L}(W_p^{2+\alpha}(\Omega)^3,W_p^{\alpha}(\Omega))$ and

$$v\to A'_{1i}(v)\quad\text{is a continuous mapping of } W_p^{2+\alpha}(\Omega)^3\quad\text{into } \mathcal{L}(W_p^{2+\alpha}(\Omega)^3,W_p^{\alpha}(\Omega)).\tag{9.31}$$

Let $v,h\in W_p^{2+\alpha}(\Omega)^3$ and $\gamma\in\mathbb{R}$, $|\gamma|$ is small. By using the mean value theorem [18], III, 5, Corollary 1, we obtain

$$\Big|\Big|\frac{A_{1i}(v+\gamma h)-A_{1i}(v)}{\gamma}-A'_{1i}(v)h\Big|\Big|_{W_p^{\alpha}(\Omega)}$$
$$\le\sup_{\xi}||A'_{1i}(v+\xi h)-A'_{1i}(v)||_{\mathcal{L}(W_p^{2+\alpha}(\Omega)^3,W_p^{\alpha}(\Omega))}||h||_{W_p^{2+\alpha}(\Omega)^3},$$
$$0<\xi<\gamma\ \text{ if }\ \gamma>0,\quad 0>\xi>\gamma\ \text{ if }\ \gamma<0.\tag{9.32}$$

By (9.31) the right-hand side of (9.32) tends to zero, when γ tends to zero. Therefore, $A'_{1i}(v)$ is the Gâteaux derivative of A_{1i} at the point v, and by (9.31) it is the Fréchet derivative, and the mapping $v\to A(v)$ is Fréchet continuously differentiable in $W_p^{2+\alpha}(\Omega)^3$.

By analogy, we prove that the function $v\to A'(v)$ is Fréchet continuously differentiable in $W_p^{2+\alpha}(\Omega)^3$, and its derivative is defined by (9.25) and (9.26).

Taking (C9.1), (9.24), (9.26), and (9.27) into account, we obtain that the functions

$$v\to||A'_1(v)||_{\mathcal{L}(W_p^{2+\alpha}(\Omega)^3,W_p^{\alpha}(\Omega)^3)},$$
$$v\to||A''_1(v)||_{\mathcal{L}_2(W_p^{2+\alpha}(\Omega)^3,W_p^{2+\alpha}(\Omega)^3;W_p^{\alpha}(\Omega)^3)}$$

are bounded in any ball of $W_p^{2+\alpha}(\Omega)^3$. It follows from (9.5) and (9.6) that $A(0)=0$.

Our lemma is proved.

Lemma 9.2. *Suppose that the conditions* (C9.1), (C9.2), *and* (9.10) *are satisfied. Then for any* $v\in W_p^{2+\alpha}(\Omega)^3$, *where* $\alpha\in(0,1)$, $\alpha p>3$, *the operator* $A'(v)$ *is invertible, and it is an isomorphism of* $W_p^{2+\alpha}(\Omega)^3$ *onto* $W_p^{\alpha}(\Omega)^3\times W_p^{2+\alpha-\frac{1}{p}}(S)^3$.

Proof. Let $v\in W_p^{2+\alpha}(\Omega)^3$ and h,y be two functions from $H^2(\Omega)^3\bigcap H_0^1(\Omega)^3$. We multiply the equation (9.24) by y_i sum over i, and integrate over Ω. Taking the Green formula and (9.7)–(9.9) into account, we obtain

$$(A'_1(v)h,y)=(A'_1(v)y,h),\tag{9.33}$$
$$(A'_1(v)h,h)\ge\frac{1}{3}\beta\int_\Omega(\operatorname{div}h)^2\,dx+\mu\int_\Omega I(h)\,dx,\tag{9.34}$$

where $\mu=\min(2a_1,2a_3)$.

(9.2)–(9.4) and (9.21) imply that

$$\int_\Omega\sum_{i,j=1}^{3}(\varepsilon_{ij}(h))^2\,dx=\int_\Omega(I(h)+\frac{1}{3}(\operatorname{div}h)^2)\,dx.\tag{9.35}$$

(9.34), (9.35), and the Korn inequality yield, see [13],

$$(A_1'(v)h, h) \geq c||h||^2_{H_0^1(\Omega)^3}, \quad c > 0. \tag{9.36}$$

We also have

$$|(A_1'(v)h, y)| \leq c_1||h||_{H_0^1(\Omega)^3)}||y||_{H_0^1(\Omega)^3)}. \tag{9.37}$$

Therefore, the operator $A_1'(v)$ is uniformly elliptic. For the Dirichlet boundary conditions (9.6), the complementing boundary condition is satisfied, see [1]. By (9.36) the kernel of $A'(v)$ contains zero only. By using the Green formula, we can see that the operator $A'(v)$ is formally selfadjoint. Therefore, the kernel of the adjoint operator $(A'(v))^*$ contains only zero element of the space $(W_p^\alpha(\Omega)^3)^* \times W_q^{-2-\alpha+\frac{1}{p}}(S)^3$, $q = \frac{p}{p-1}$.

Now it follows from [17], [19] that for any $(K, q) \in W_p^\alpha(\Omega)^3 \times W_p^{2+\alpha-\frac{1}{p}}(S)^3$, there exists a unique $w \in W_p^{2+\alpha}(\Omega)^3$ such that $A'(v)w = (K, q)$. For any $v \in W_p^{2+\alpha}(\Omega)^3$, the operator $A'(v)$ is an isomorphism of $W_p^{2+\alpha}(\Omega)^3$ onto $W_p^\alpha(\Omega)^3 \times W_p^{2+\alpha-\frac{1}{p}}(S)^3$. $\square$

9.4. **Approximation of the solution to the problem** (9.5), (9.6)**.** The ensuing theorem follows from theorems 8.1, 9.1 and Lemmas 9.1 and 9.2.

Theorem 9.2. *Suppose that the conditions* (C9.1), (C9.2), *and* (9.10) *are satisfied. Let also* $(K, q) \in W_p^\alpha(\Omega)^3 \times W_p^{2+\alpha-\frac{1}{p}}(S)^3$, *where* $\alpha \in (0, 1)$, $\alpha p > 3$. *Then there exists a unique* $u \in W_p^{2+\alpha}(\Omega)^3$ *that satisfies* (9.5) *and* (9.6), *and the operator* A *is a homomorphism of* $W_p^{2+\alpha}(\Omega)^3$ *onto* $W_p^\alpha(\Omega)^3 \times W_p^{2+\alpha-\frac{1}{p}}(S)^3$.

Let $\{U_k\}$ be a sequence of finite dimensional subspaces of $W_p^{2+\alpha}(\Omega)^3$ such that

$$\lim_{k\to\infty} \inf_{h\in U_k} ||v - h||_{W_p^{2+\alpha}(\Omega)^3} = 0, \quad v \in W_p^{2+\alpha}(\Omega)^3. \tag{9.38}$$

A sequence of subspaces U_k, which satisfy the condition (9.38), can be constructed in the manner presented in 6.1. In this case we use cubic splines which are twice continuously differentiable.

Let u_k be a sequence of solutions to the problem

$$u_k \in U_k, \quad P_{kl}A(u_k) = P_{kl}(K, q), \quad k \in \mathbb{N}, \tag{9.39}$$

where P_{kl} in the projection specified in Subsection 8.2.

Theorem 9.3. *Suppose that the conditions* (C9.1), (C9.2), (9.10) *and* (9.38) *are satisfied. Let also* $(K, q) \in W_p^\alpha(\Omega)^3 \times W_p^{2+\alpha-\frac{1}{p}}(S)^3$, *where* $\alpha \in (0, 1)$, $\alpha p > 3$. *Let also* u_k *be a sequence of solutions to the problem* (9.39). *Then* $u_k \to u$ *in* $W_p^{2+\alpha_1}(\Omega)^3$, *where* u *is the solution to the problem* (9.5), (9.6), *and* $\alpha_1 \in (0, \alpha)$.

Indeed, the embedding of $W_p^\alpha(\Omega)$ into $W_p^{\alpha_1}(\Omega)$ and $W_p^{2+\alpha-\frac{1}{p}}(S)$ into $W_p^{2+\alpha_1-\frac{1}{p}}(S)$ are compact at $\alpha_1 \in (0, \alpha)$. Bearing in mind Lemmas 9.1 and 9.2, we obtain Theorem 9.3 by applying Theorem 8.2.

Remark. By analogy with the above, we can construct approximations of smooth solutions of nonlinear operators, which are restrictions of strongly monotone operators, such that, they are twice Fréchet continuously differentiable in corresponding spaces of smooth functions, and their first derivatives are invertible.

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